A remark on the $(2,2)$-domination number

Torsten Korneffel, Dirk Meierling and Lutz Volkmann

Lehrstuhl II für Mathematik, RWTH Aachen University, 52056 Aachen, Germany
e-mail: {korneffe,meierling,volkm}@math2.rwth-aachen.de

Abstract

A subset D of the vertex set of a graph G is a (k,p)-dominating set if every vertex $v \in V(G) \setminus D$ is within distance k to at least p vertices in D. The parameter $\gamma_{k,p}(G)$ denotes the minimum cardinality of a (k,p)-dominating set of G. In 1994, Bean, Henning and Swart posed the conjecture that $\gamma_{k,p}(G) \leq \frac{p}{p+k}n(G)$ for any graph G with $\delta_k(G) \geq k+p-1$, where the latter means that every vertex is within distance k to at least $k+p-1$ vertices other than itself. In 2005, Fischermann and Volkmann confirmed this conjecture for all integers k and p for the case that p is a multiple of k. In this paper we show that $\gamma_{2,2}(G) \leq (n(G)+1)/2$ for all connected graphs G and characterize all connected graphs with $\gamma_{2,2} = (n+1)/2$. This means that for $k = p = 2$ we characterize all connected graphs for which the conjecture is true without the precondition that $\delta_2 \geq 3$.

Keywords: domination, distance domination number, p-domination number.
2000 Mathematics Subject Classification: 05C69.

1 Terminology and introduction

In this paper we consider simple, finite and undirected graphs $G = (V,E)$ with vertex set V and edge set E. The number of vertices $|V|$ is called the *order* of G and is denoted by $n(G)$.

If there is an edge between two vertices $u, v \in V$, then we denote the edge by uv. Furthermore, we call the vertex v a *neighbor* of u and say that uv is incident with u. The *neighborhood* of a vertex u is defined as the set $\{v \mid uv \in E\}$ and is usually denoted by $N(u)$. For a vertex $v \in V$ we define the *degree* of v as $d(v) = |N(v)|$. If $d(v) = 1$, then the vertex v is called a *leaf* of G. The *minimum degree* of G is denoted by $\delta(G) = \min\{d(v) \mid v \in V(G)\}$.

For any positive integer k and any graph G the *k-th power* G^k of G is the graph with vertex set $V(G)$ where two different vertices are adjacent if and only if the distance between them is at most k in G. Furthermore, the *minimum k-degree* $\delta_k(G)$ of G is defined by $\delta_k(G) = \delta(G^k)$.

Let $X \subseteq V$ be a subset of the vertex set of a graph $G = (V,E)$. Then $G - X$ denotes the graph that is obtained by removing all vertices of X and all edges that are incident with at least one vertex of X from G. The *diameter* of a graph is defined as the maximum distance between all pairs of vertices.

For two positive integers k and p a subset D of the vertex of a graph G is a (k,p)-*dominating set* of G if every vertex $v \in V(G) \setminus D$ is within distance k to at least p vertices in D. The parameter $\gamma_{k,p}(G)$ denotes the minimum cardinality of a (k,p)-dominating set of G and is called the (k,p)-*domination number*.

This domination concept is a generalization of the two concepts *distance domination* and p-*domination*. For $p = 1$ a (k,p)-dominating set of G is called a *distance-k dominating set* and for $k = 1$ a (k,p)-dominating set of G is called a p-*dominating set*.

For other graph terminologies we refer the reader to the monographs by Haynes, Hedetniemi and Slater [4, 5].

In 1994, Bean, Henning and Swart [1] posed the following conjecture for the (k,p)-domination number $\gamma_{k,p}$.

Conjecture 1 (Bean, Henning & Swart [1] 1994)**.** *Let k and p be arbitrary positive integers and let G be a graph of minimum k-degree $\delta_k(G) \geq k + p - 1$. Then*

$$\gamma_{k,p}(G) \leq \frac{p}{p+k} n(G).$$

This conjecture is valid for $p = 1$ and all integers $k \geq 1$ as proved by Meir and Moon [6] in 1975 (the distance-k domination number is called k-*covering number* in [6]). The conjecture is also true for $k = 1$ and all integers $p \geq 1$ as proved by Cockayne, Gamble and Shepherd [2] in 1985. In 2005, Fischermann and Volkmann [3] confirmed that the conjecture is valid for all integers k and p, where p is a multiple of k, and presented weaker statements in the remaining cases.

Note that if $k = p = 2$, then Conjecture 1 requires that $\delta_2(G) \geq 3$. In this paper, we shall show that the conjecture is true for $k = p = 2$ without the precondition that $\delta_2(G) \geq 3$ for all connected graphs with the exception of the following class.

Definition 2. *A* spider *is a graph G with vertex set $V = \{x\} \cup \{y_i \mid i = 1, 2, \ldots, k\} \cup \{z_i \mid i = 1, 2, \ldots, k\}$ and edge set $E = \{xy_i \mid i = 1, 2, \ldots, k\} \cup \{y_i z_i \mid i = 1, 2, \ldots, k\}$, where $k \geq 1$ is an integer. The vertex x is called the* centre *of G.*

In particular, note that if G is a spider, then $\delta_2(G) = 2$. We can calculate the $(2,2)$-domination number of spiders as follows.

Theorem 3. *If G is a spider with n vertices, then $\gamma_{2,2}(G) = \frac{n+1}{2}$.*

Proof. Let G be a spider as defined in Definition 2. Then it is easy to see that $\{x\} \cup \{y_i \mid i = 1, 2, \ldots, k\}$ is a $(2,2)$-dominating set of G.

It remains to proof that there exists no $(2,2)$-dominating set D of G such that $|D| < \frac{n+1}{2}$. Assume to the contrary that D is a $(2,2)$-dominating set of G such that $|D| < \frac{n+1}{2}$. Note that for each pair y_i, z_i of vertices of G the vertex y_i or the vertex z_i or both belong to D. Since $|D| < \frac{n+1}{2}$, it follows that $|D \cap \{y_i, z_i\}| = 1$ for each $i = 1, 2, \ldots, k$. If $D = \{z_1, z_2, \ldots, z_k\}$, then y_1 is not $(2,2)$-dominated by D, a contradiction. Otherwise let i be an integer such that $y_i \in D$. But then z_i is not $(2,2)$-dominated by D, again a contradiction. This completes the proof of this theorem. $\square$

To proof our main result we need the following graph operations.

Definition 4. *Let G be a connected graph and let x be a vertex of G.*

(i) The graph G_x is obtained from G by adding two leaves as neighbors to x, i.e., $V(G_x) = V(G) \cup \{y, z\}$ and $E(G_x) = E(G) \cup \{xy, xz\}$.

(ii) The graph G^x is obtained from G by adding a path yz of length 1 to G such that y is a neighbor of x, i.e., $V(G^x) = V(G) \cup \{y, z\}$ and $E(G^x) = E(G) \cup \{xy, yz\}$.

2 Results

First we proof a structural result.

Theorem 5. *Let G be a connected graph and let D be a $(1,1)$- and $(2,2)$-dominating set of G. If x is an arbitrary vertex of G, then either $D \cup \{x\}$ or $D \cup \{y\}$ is a $(1,1)$- and $(2,2)$-dominating set of G_x and $D \cup \{z\}$ is a $(1,1)$- and $(2,2)$-dominating set of G^x.*

Proof. Let x be an arbitrary vertex of G and let D be a $(1,1)$- and $(2,2)$-dominating set of G.

We consider G^x first. If $x \in D$, then both neighbors of y in G^x belong to $D \cup \{z\}$. Otherwise x has a neighbor $v \in D$ which naturally has distance 2 from y. Therefore $D \cup \{z\}$ is a $(1,1)$- and $(2,2)$-dominating set of G^x.

Now we consider G_x. If $x \in D$, then, since z is a neighbor of x and has distance 2 from y, the set $D \cup \{y\}$ is a $(1,1)$- and $(2,2)$-dominating set of G_x. Otherwise x has a neighbor $v \in D$ which naturally has distance 2 from y and z. Therefore $D \cup \{x\}$ is a $(1,1)$- and $(2,2)$-dominating set of G_x. $\square$

Our main result follows.

Theorem 6. *If T is a tree on $n \geq 3$ vertices, then there exists a minimum $(1,1)$- and $(2,2)$-dominating set D of T such that $|D| \leq \frac{n+1}{2}$. In addition, equality holds if and only if T is a spider.*

Proof. We shall prove the proposition by induction on n.

The only tree T with $n = 3$ vertices is the path xyz of length 2. This means that T is a spider and two arbitrary vertices of T are a $(1,1)$- and $(2,2)$-dominating set of T.

If T is a tree with $n = 4$ vertices, then either T is the path of length 3 or T is a star. In the first case the two leaves of T and in the latter case the centre of T and an arbitrary other vertex are a $(1,1)$- and $(2,2)$-dominating set of T.

Let T be a tree on $n = 5$ vertices. If T is the path $v_1v_2v_3v_4v_5$ of length 4, then T is a spider and $\{v_1, v_3, v_5\}$ is a $(1,1)$- and $(2,2)$-dominating set of T. If T has diameter 3, then the two vertices that are not leaves form a $(1,1)$- and $(2,2)$-dominating set of T. In the remaining case T has diameter 2 and thus, T is a star. Then the centre of T and another arbitrary vertex of T form a $(1,1)$- and $(2,2)$-dominating set of T.

Now let T be a tree on $n \geq 6$ vertices. Note that each spider has an odd number of vertices. In addition, note that there exists a vertex x in T such that either

(1) two leaves y, z of T are neighbors of x or

(2) the vertex x is not a leaf and there exists a vertex y with $d(y) = 2$ that has x and a leaf z as neighbors.

Let x, y, z be vertices of T that fulfill either (1) or (2). By the induction hypothesis, the tree $T - \{y, z\}$ has a minimum $(1, 1)$- and $(2, 2)$-dominating set D such that

$$|D| \leq \frac{n(T - \{y, z\}) + 1}{2} = \frac{n - 1}{2}.$$

If x, y, z fulfill (1), then, by Theorem 5, $D \cup \{x\}$ or $D \cup \{y\}$ is a $(1, 1)$- and $(2, 2)$-dominating set of $T = (T - \{y, z\})_x$. If x, y, z fulfill (2), then, by Theorem 5, $D \cup \{z\}$ is a $(1, 1)$- and $(2, 2)$-dominating set of $T = (T - \{y, z\})^x$.

If $T - \{y, z\}$ is not a spider in one of the cases above, then, by the induction hypothesis, $|D| \leq \frac{n-2}{2}$ and thus,

$$|D \cup \{x\}| \leq |D \cup \{y\}| = |D \cup \{z\}| = |D| + 1 \leq \frac{n}{2}.$$

Suppose now that $T - \{y, z\}$ is a spider for all vertices x, y, z that fulfill (1) or (2). In this case we shall show that T itself is a spider or a path P_7 of order 7 which has a $(1, 1)$- and $(2, 2)$-dominating set of size 3. Let $T - \{y, z\}$ be a spider as defined in Definition 2.

Assume that x, y, z fulfill (1). Then there exists an integer i such that $T - \{y_i, z_i\}$ is not a spider, a contradiction.

So assume now that x, y, z fulfill (2). Note that $k \geq 2$, since $|V(T)| \geq 6$.

If $k \geq 3$ or $k = 2$ and $T \neq P_7$, then either there exists an integer i such that $T - \{y_i, z_i\}$ is not a spider, again a contradiction, or the centre of T is the only neighbor of y in T. But in the latter case it is immediate that T is a spider.

If $k = 2$ and $T = P_7$, then let $T = v_1 v_2 \ldots v_7$. In this case $\{v_1, v_4, v_7\}$ is a $(1, 1)$- and $(2, 2)$-dominating set of T, which completes the proof of this theorem. □

Theorem 6 immediately implies the following corollaries.

Corollary 7. *If T is a tree on $n \geq 3$ vertices, then $\gamma_{2,2}(T) \leq \frac{n+1}{2}$ with equality if and only if T is a spider.*

Corollary 8. *If G is a connected graph on $n \geq 3$ vertices, then there exists a minimum $(1, 1)$- and $(2, 2)$-dominating set D of G such that $|D| \leq \frac{n+1}{2}$. In addition, equality holds if and only if G is a spider.*

Proof. If G has a spanning tree that is not a spider, then the inequality is true by Theorem 6. Otherwise either G itself is a spider or G is a cycle $v_1 v_2 v_3 v_4 v_5 v_1$ of length 5. In the latter case $\{v_1, v_3\}$ is a $(1, 1)$- and $(2, 2)$-dominating set of G with the required cardinality. □

Corollary 9. *If G is a connected graph on $n \geq 3$ vertices, then $\gamma_{2,2}(G) \leq \frac{n+1}{2}$ with equality if and only if G is a spider.*

References

[1] T.J. Bean, M.A. Henning and H.C. Swart: On the integrity of distance domination in graphs. *Australas. J. Comb.* **10** (1994), 29–43.

[2] E.J. Cockayne, B. Gamble and B. Shepherd: An upper bound for the k-domination number of a graph. *J. Graph Theory* **9** (1985), 101–102.

[3] M. Fischermann and L. Volkmann: A remark on a conjecture for the (k, p)-domination number. *Utilitas Mathematica* **67** (2005), 223–227.

[4] T.W. Haynes, S.T. Hedetniemi and P.J. Slater: Fundamentals of domination in graphs. *Marcel Dekker, Inc., New York* (1998).

[5] T.W. Haynes, S.T. Hedetniemi and P.J. Slater: Domination in graphs, Advanced Topics. *Marcel Dekker, Inc., New York* (1998).

[6] A. Meir and J.W. Moon: Relations between packing and covering numbers of a tree, *Pacific J. Math.* **61** (1975), no. 1, 225–233.