Cycles in Complementary Prisms

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Abstract

The complementary prism $G\overline{G}$ of a graph G arises from the disjoint union of G and the complement $\overline{G}$ of G by adding a perfect matching joining corresponding pairs of vertices in G and $\overline{G}$. Partially answering a question posed by Haynes et al. (The complementary product of two graphs, Bull. Inst. Comb. Appl. 51, 21-30, 2007) we provide an efficient characterization of the circumference of the complementary prism $T\overline{T}$ of a tree T and show that $T\overline{T}$ has cycles of all lengths between 3 and its circumference. Furthermore, we prove that for a given graph of bounded maximum degree it can be decided in polynomial time whether its complementary prism is Hamiltonian.

Keywords: Complementary prism; prism; Hamiltonian; pancyclic

MSC2010: 05C38, 05C45, 05C76, 05C70.

1 Introduction

In [9] Haynes et al. introduce the *complementary product* of two graphs as a generalization of the Cartesian product. As an interesting special case they consider the *complementary prism* $G\overline{G}$ of a graph G. The complementary prism $G\overline{G}$ is isomorphic to the graph that arises from the disjoint union of G and the complement $\overline{G}$ of G by adding a perfect matching joining corresponding pairs of vertices in G and $\overline{G}$.

In [9] Haynes et al. study elementary properties of complementary products/prisms related to degrees, distances, independence, and domination. Complementary prisms were further investigated in [3, 4, 11, 12]. At the end of [9] Haynes et al. list some open questions and problems. The first of their questions is which complementary prisms are Hamiltonian.

Hamiltonicity and more generally cycles in ordinary prisms, that is, Cartesian products with K_2, attracted a lot of interest. Kaiser et al. [10] trace this interest back to Barnette's famous open conjecture that the graphs of simple 4-polytopes are Hamiltonian. A related open conjecture posed by Rosenfeld and Barnette [15] is that the prism of a 3-connected planar graph is Hamiltonian. Hamiltonicity and pancyclism of prisms were studied for instance in [1,2,6,7,14], see [10] and Section 30.2 of [8] for further discussion of the background and partial results.

In the present paper we study Hamiltonicity of complementary prisms and provide some partial answers to the question posed in [9]. We characterize the circumference of complementary prisms of trees, which implies a characterization of those trees whose complementary prism is Hamiltonian. Furthermore, we study the algorithmic problem of deciding whether the complementary prism of a given graph is Hamiltonian.

We consider finite, simple, and undirected graphs and use standard terminology and notation. Let G be a graph. If u is a vertex of G, then $\overline{u}$ denotes the vertex of $\overline{G}$ corresponding to u. Therefore, if the vertex set $V(G)$ of G is $\{v_1, \ldots, v_n\}$, then the vertex set of $G\overline{G}$ will be denoted by $V(G) \cup \overline{V}(G)$ where $\overline{V}(G) = \{\overline{v}_1, \ldots, \overline{v}_n\}$. Furthermore, the graph $G\overline{G}$ contains an edge between two distinct of its vertices x and y if and only if either $x, y \in V(G)$ and $xy \in E(G)$ or $x, y \in \overline{V}(G)$ and $xy \notin E(G)$ or there is some vertex u of G with $\{x, y\} = \{u, \overline{u}\}$. A graph G is *c-pancyclic* for a positive integer c if G has cycles of all lengths ℓ with $3 \leq \ell \leq c$. The maximum length of a cycle of a graph is its *circumference*.

2 Trees

Our first result is a lower bound on the number of cycle lengths of complementary prisms of trees. A cycle in the complementary prism $T\overline{T}$ of a tree T is *good* if it uses two independent edges between vertices in $\overline{V}(T)$ and *very good* if it uses three independent edges between vertices in $\overline{V}(T)$. The reason to consider good cycles in $T\overline{T}$ is that they can easily be extended.

Theorem 1. *If T is a tree of order $n \geq 5$, then $T\overline{T}$ is $(n+2)$-pancyclic. Furthermore, $T\overline{T}$ has circumference $n+2$ if and only if $T = K_{1,n-1}$.*

Proof. First, we assume that $T = K_{1,n-1}$. The complement $\overline{T}$ of T consists of a clique of order $n-1$ and an isolated vertex. Hence $\overline{T}$ is $(n-1)$-pancyclic. Let $v, w \in V(T)$ be two distinct leaves of T and let vuw the unique v-w-path in T. Note that there exist paths of lengths $1, 2, \ldots, n-2$ between $\overline{v}$ and $\overline{w}$ in $\overline{T}$. These paths together with the path $\overline{v}vuw\overline{w}$ form cycles of lengths $5, 6, \ldots, n+2$ in $T\overline{T}$, that is, $T\overline{T}$ is $(n+2)$-pancyclic. Clearly, the isolated vertex of $\overline{T}$ has degree 1 in $T\overline{T}$ and is therefore not contained in any cycle of $T\overline{T}$. Furthermore, since every leaf of T has degree exactly 2 in $T\overline{T}$, every cycle in $T\overline{T}$ contains at most two leaves of T and therefore at most three vertices of T. It follows that the circumference of $T\overline{T}$ is exactly $n+2$.

Next, we assume that $T \neq K_{1,n-1}$. By induction on n, we show that $T\overline{T}$ is $(n+3)$-pancyclic and has a good cycle of length $n+3$.

If $n = 5$, then T is either P_5 or the tree $K'_{1,3}$ that arises from a claw $K_{1,3}$ by subdividing one edge. We denote the vertices as in Figure 1. If T is P_5, then $\overline{v}_1\overline{v}_3\overline{v}_5\overline{v}_1$, $\overline{v}_1\overline{v}_4\overline{v}_2\overline{v}_5\overline{v}_1$, $\overline{v}_1\overline{v}_4\overline{v}_2\overline{v}_5\overline{v}_3\overline{v}_1$, $v_1v_2v_3v_4\overline{v}_4\overline{v}_1v_1$, and $v_1v_2v_3v_4v_5\overline{v}_5\overline{v}_1v_1$ are cycles of lengths between 3 and 7 and $v_1v_2\overline{v}_2\overline{v}_4v_4v_3\overline{v}_3\overline{v}_1v_1$ is a good cycle of length 8. If T is $K'_{1,3}$, then $\overline{v}_1\overline{v}_4\overline{v}_5\overline{v}_1$, $\overline{v}_1\overline{v}_5\overline{v}_2\overline{v}_4\overline{v}_1$, $v_1v_2v_3\overline{v}_3\overline{v}_1v_1$, $v_1v_2v_3v_4\overline{v}_4\overline{v}_1v_1$, and $v_1v_2v_3v_4\overline{v}_4\overline{v}_5\overline{v}_1v_1$ are cycles of lengths between 3 and 7 and $v_1v_2v_3v_4\overline{v}_4\overline{v}_2\overline{v}_5\overline{v}_1v_1$ is a good cycle of length 8.

Now let $n \geq 6$. Since $T \neq K_{1,n-1}$, there exists a leaf v of T such that $T - v \neq K_{1,n-2}$. Let

$T' = T - v$. By induction, $T'\overline{T'}$ is $(n+2)$-pancyclic and has a good cycle C of length $n+2$. Since $\overline{v}$ has only one non-neighbor in $\overline{V}(T')$, the cycle C contains an edge $\overline{v}_a\overline{v}_b$ such that $\overline{v}_a$ and $\overline{v}_b$ are neighbors of $\overline{v}$. Replacing $\overline{v}_a\overline{v}_b$ with $\overline{v}_a\overline{v}\,\overline{v}_b$ results in a good cycle of length $n+3$ in $T\overline{T}$, which completes the proof. $\square$

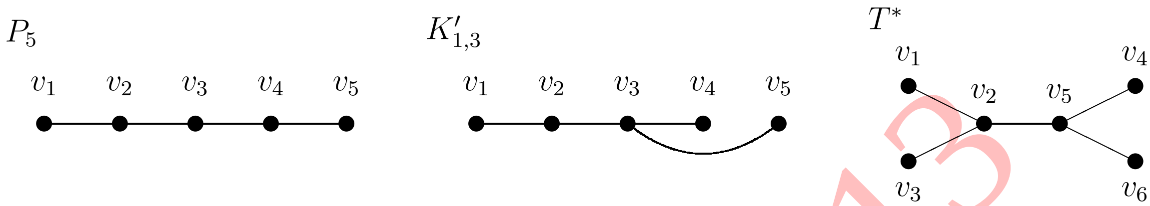

Figure 1: The path P_5, the tree $K'_{1,3}$, and the tree T^*.

In Theorem 1, we saw that complementary prisms of trees can be far from being Hamiltonian. Our main result in this section characterizes the circumference of these graphs. In fact, for a tree T, the circumference of $T\overline{T}$ is determined by a unique substructure of T that forces local restrictions on cycles of $T\overline{T}$.

Let T be a tree. A subforest F of T is a *special subforest of T generated by U* if U is a set of vertices of T and F has components $F_1, \ldots, F_t$ such that for $i \in [t]$, $U_i = U \cap V(F_i)$, and $N_i = N_T(U_i)$, we have that

- F_i is the subtree of T induced by $U_i \cup N_i$,
- U_i and N_i form the bipartition of the tree F_i, and
- every vertex in N_i has degree at least 3 in F_i.

Let $N = N_T(U) = N_1 \cup \cdots \cup N_t$. Note that every vertex in U has the same degree in F as in T. Since all leaves of F necessarily belong to U, the forest F uniquely determines the set U. While U is an independent set of T, the set N is independent in F but not necessarily independent in T.

The *deficit* $\mathrm{def}(F)$ of F is defined as

$$\mathrm{def}(F) = |U| - 2|N|.$$

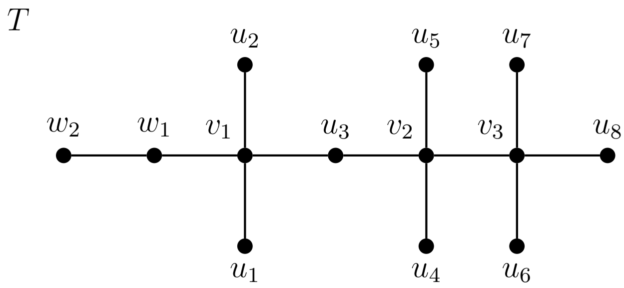

Figure 2: The maximum special subforest F of T is generated by $\{u_1, \ldots, u_8\}$. It consists of the two components F_1 generated by $\{u_1, \ldots, u_5\}$ and F_2 generated by $\{u_6, u_7, u_8\}$. The edge v_2v_3 belongs to T but not to F. The vertices w_1 and w_2 do not belong to any special subforest. The deficit of F is 2.

If F has t components, then it has exactly $|V(F)| - t = |U| + |N| - t$ edges. By the degree condition on the vertices in N, the forest F has at least $3|N|$ edges. This implies that the deficit of F is at least t, that is, every non-empty special forest has positive deficit.

As we will see below in Lemma 7, every tree T has a unique special subforest F of maximum order. Therefore,

$$f(T) = 2|V(T)| - \mathrm{def}(F)$$

is a well defined quantity depending only on T.

Our main result is the following.

Theorem 2. *If T is a tree of order $n \geq 6$ and diameter at least 4, then the circumference of $T\overline{T}$ is $f(T)$ and $T\overline{T}$ is $f(T)$-pancyclic.*

In view of Theorem 2, some remarks concerning trees of small order or of diameter at most 3 are in order. Theorem 1 already completely describes the cycle lengths of complementary prisms of stars. It is instructive to check that P_4 is the only tree of order at most 5 with a Hamiltonian complementary prism. Also the cycle lengths of complementary prisms of trees of diameter 3 are easily determined. The tree T^* of order 6 and diameter 3 (cf. Figure 1) shows that Theorem 2 does not hold for all trees. In fact, $f(T^*) = 12$ while $\overline{v}_1\overline{v}_5\overline{v}_3v_3v_2\overline{v}_2\overline{v}_6v_6v_5v_4\overline{v}_4\overline{v}_1$ is a longest cycle of $T^*\overline{T^*}$ of length 11.

Since every non-empty special subforest has positive deficit, Theorem 2 yields a characterization of those trees whose complementary prism is Hamiltonian.

Corollary 3. *If T is a tree of order $n \geq 6$ and diameter at least 4, then $T\overline{T}$ is Hamiltonian if and only if T does not contain a non-empty special subforest.*

The next three lemmas collect essential properties of special subforests.

Lemma 4. *If T is a tree of order n and F is a special subforest of T, then the circumference of $T\overline{T}$ is at most $2n - \mathrm{def}(F)$.*

Proof. Let F be generated by U. Let $N = N_T(U)$. Let C be a cycle of maximum length in $T\overline{T}$. Let $U^* = U \cap V(C)$. Since every vertex u in U^* has exactly one neighbor in $\overline{V}(T)$, every such vertex is joined by some edge of C to some neighbor $p(u)$ in N. Note that there can be two choices for $p(u)$ and that, for every vertex w in N, there are at most two vertices u and v in U^* with $w = p(u) = p(v)$. Double counting the pairs $(u, p(u))$ where $u \in U^*$ implies $|U^*| \leq 2|N|$. Since the length of C is at most

$$|\overline{V}(T)| + |V(T)| - (|U| - |U^*|) \leq |\overline{V}(T)| + |V(T)| - (|U| - 2|N|) = 2n - \mathrm{def}(F),$$

the proof is complete. □

Lemma 5. *Let T be a tree and let F_i be a special subforest of T generated by U_i for $i \in \{1, 2\}$. Then $F_1 \cup F_2$ is a special subforest of T generated by $U_1 \cup U_2$.*

Proof. Let $N_i = N_T(U_i)$ for $i \in \{1, 2\}$. For a contradiction, we assume that $U_1 \cap N_2$ is not empty. Let $u \in U_1 \cap N_2$. Since $u \in N_2$, the vertex u has at least 3 neighbors in F_2 that all belong to U_2. Since $u \in U_1$, all neighbors of u in F_2 also belong to N_1. Hence every vertex in $U_1 \cap N_2$ has at least 3 neighbors in $U_2 \cap N_1$ and, by symmetry, every vertex in $U_2 \cap N_1$ has at least 3 neighbors in $U_1 \cap N_2$. Hence $F_1 \cap F_2$ has a component of minimum degree at least 3, which is impossible because T is a tree. We obtain that $(U_1 \cap N_2) \cup (U_2 \cap N_1)$ is empty, that is, $U = U_1 \cup U_2$ is independent. Let $N = N_T(U)$ and let F be the subgraph of T induced by the edges in F_1 and F_2. Since $(U_1 \cap N_2) \cup (U_2 \cap N_1)$ is empty and $N = N_1 \cup N_2$, U and N are the two partite sets of F, and every vertex in N has degree at least 3 in F. Hence $F = F_1 \cup F_2$ is the special subforest of T generated by U. □

Lemma 6. *If F and F' are special subforests of a tree T with $V(F) = V(F')$, then $F = F'$.*

Proof. We prove the statement by induction on the order n of T. For $n \leq 3$, the tree T has no non-empty special subforest and the statement is trivially true. Now let $n \geq 4$. Let F and F' be special subforests of T with the same vertex set V. Let F and F' be generated by U and U', respectively. Let $N = N_T(U)$ and $N' = N_T(U')$, that is, $V = U \cup N = U' \cup N'$.

Let u be an endvertex of a longest path P in F. Since T is a tree of order at least 4, the vertex u is not isolated in T. Hence u also has a neighbor v in F. By the choice of P, the vertex u is a leaf of F. This implies that u is a leaf of T and hence also of F'. We obtain that $u \in U \cap U'$ and $v \in N \cap N'$. Let X and X' denote the sets of leaves of F and F' that are adjacent to v, respectively. Since F and F' are special subforests, it follows that $X = X'$, $|X| \geq 2$, and all vertices in X are leaves of T. Furthermore, if P has length 2, then $|X| \geq 3$. Note that every neighbor of v in F or F' belongs to U or U', respectively. It follows that $F - (\{v\} \cup X)$ and $F' - (\{v\} \cup X)$ are special subforests of $T - (\{v\} \cup X)$ with the same vertex set. By induction, $F - (\{v\} \cup X) = F' - (\{v\} \cup X)$. If v has a neighbor w in F, then $w \in U$. By induction, it follows that $w \in U'$, which implies that w is also a neighbor of v in F'. By symmetry, this implies that v has the same neighborhood in F as in F'. Hence $F = F'$ and the proof is complete. $\square$

Lemma 7. *Every tree has a unique special subforest of maximum order.*

Proof. Let T be a tree. By Lemma 5, the union of two special subforests of T is a special subforest of T. This implies that all special subforests of maximum order of T have the same vertex set. Now Lemma 6 implies that all special subforests of maximum order of T coincide. $\square$

In order to determine the circumference of the complementary prism of a tree, it suffices to determine its unique special subforest of maximum order. The following proposition allows to do so recursively in polynomial time.

Proposition 8. *Let T be a tree of order at least 2. Let F be the special subforest of T of maximum order. Let v be a vertex of T that is adjacent to the maximum possible number ℓ of leaves of T. Let W denote the set of leaves of T adjacent to v. Let $T_1, \ldots, T_t$ be the components* *of $T-(\{v\}\cup W)$. For $i\in[t]$, let u_i be the neighbor of v in $V(T_i)$, let F_i be the special subforest of T_i of maximum order, and let F_i be generated by U_i.*

Given these assumptions,

$$F=\begin{cases} T[\{v\}\cup W]\cup F_1\cup\cdots\cup F_t & \text{, if } (\ell\geq 3)\vee(\ell=2\wedge\exists i\in[t]:u_i\in U_i),\\ F_1\cup\cdots\cup F_t & \text{, if } \ell=2\wedge\forall i\in[t]:u_i\notin U_i,\\ \emptyset & \text{, if } \ell=1.\end{cases}$$

Proof. Let F be generated by U. Let $N=N_T(U)$ and, for $i\in[t]$, let $N_i=N_{T_i}(U_i)$. Let $n=|V(F)|$ and, for $i\in[t]$, let $n_i=|V(F_i)|$.

First we assume that either $\ell\geq 3$ or $\ell=2$ and there is some index $i\in[t]$ with $u_i\in U_i$. Since $T[\{v\}\cup W]$ is a special subforest of T, Lemmas 5 and 7 imply $W\subseteq U$ and $v\in N$. Clearly, $T_i\cap F$ is a special subforest of T_i for every $i\in[t]$. Furthermore, $W\cup U_1\cup\cdots\cup U_t$ generates the special subforest $F'=T[\{v\}\cup W]\cup F_1\cup\cdots\cup F_t$ of T. These two remarks imply $n=1+\ell+n_1+\cdots+n_t$, and hence F' has maximum order, that is, $F=F'$.

Next we assume that $\ell=2$ and that there is no index $i\in[t]$ with $u_i\in U_i$. Since every vertex in N has at least three neighbors in U, this implies that $V(F)$ does not intersect $\{v\}\cup W$. Again $T_i\cap F$ is a special subforest of T_i for every $i\in[t]$. Since $F''=F_1\cup\cdots\cup F_t$ is a special subforest of T, we obtain $n=n_1+\cdots+n_t$. Hence F'' has maximum order, that is, $F=F''$.

If F is non-empty, then the definition of special subforests implies that the neighbor v' of an endvertex of a longest path in F is adjacent to at least two leaves of T. Therefore, if no vertex of T is adjacent to at least two leaves of T, then F is empty.

This completes the proof. □

Corollary 9. *There is a polynomial time algorithm that determines the circumference of the complementary prism of a given tree.*

We now proceed to the proof of Theorem 2.

Proof of Theorem 2. We say that a 5-tuple (T,F,U,C^-,C) is *good* if

(i) T is a tree of order at least 6 and diameter at least 4,

(ii) F is the unique special subforest of T of maximum order and F is generated by U,

(iii) $T\overline{T}$ is $f(T)$-pancyclic,

(iv) C^- is a good cycle of $T\overline{T}$ of lengths $f(T)-1$,

(v) C is a very good cycle of $T\overline{T}$ of lengths $f(T)$,

(vi) C contains all vertices in $(V(T)\setminus V(F))\cup\overline{V}(T)$, and

(vii) for every vertex v in $N_T(U)$, the cycle C contains a path $\overline{w}_1w_1vw_2\overline{w}_2$ where w_1 and w_2 are two neighbors of v in F.

Let T be a tree of order n and diameter at least 4 where $n\geq 6$. Let F be the unique special subforest of T of maximum order. Let F be generated by U and let $N=N_T(U)$. By Lemma 4, the circumference of $T\overline{T}$ is at most $f(T)$. Therefore, it suffices to show the existence of two cycles C^- and C of $T\overline{T}$ such that (T,F,U,C^-,C) is good. We will establish the existence of C^- and C by induction on n.

First let $n=6$. Since the diameter of T is at least 4, the diameter is 4 or 5 and F is empty, that is, $f(T)=2n=12$. We may assume that P_5 as in Figure 1 is a subgraph of T. We have observed in the proof of Theorem 1 that $P_5\overline{P}_5$ is 8-pancyclic and has a good cycle of length 8. Furthermore, $v_1v_2v_3v_4v_5\overline{v}_5\overline{v}_2\overline{v}_4\overline{v}_1v_1$ (cf. Figure 1) is a good cycle of $P_5\overline{P}_5$ of length 9. This already implies that $T\overline{T}$ is 9-pancyclic. Let v_6 denote the sixth vertex of T. By symmetry, we may assume that v_6 has a neighbor in $\{v_3,v_4,v_5\}$.

If $v_3v_6\in E(T)$ or $v_4v_6\in E(T)$ or $v_5v_6\in E(T)$, then

$$v_1v_2v_3v_4v_5\overline{v}_5\overline{v}_2\overline{v}_4\overline{v}_6\overline{v}_1v_1,\quad v_1v_2\overline{v}_2\overline{v}_5v_5v_4v_3v_6\overline{v}_6\overline{v}_4\overline{v}_1v_1,\ \text{and}\quad v_1v_2\overline{v}_2\overline{v}_4v_4v_5\overline{v}_5\overline{v}_3v_3v_6\overline{v}_6\overline{v}_1v_1;\ \text{or}$$

$$v_1v_2v_3v_4v_5\overline{v}_5\overline{v}_2\overline{v}_6\overline{v}_3\overline{v}_1v_1,\quad v_1v_2v_3v_4v_5\overline{v}_5\overline{v}_3\overline{v}_6\overline{v}_2\overline{v}_4\overline{v}_1v_1,\ \text{and}\quad v_1v_2v_3\overline{v}_3\overline{v}_5v_5v_4v_6\overline{v}_6\overline{v}_2\overline{v}_4\overline{v}_1v_1;\ \text{or}$$

$$v_1v_2v_3v_4v_5v_6\overline{v}_6\overline{v}_2\overline{v}_5\overline{v}_1v_1,\quad v_1v_2v_3v_4v_5v_6\overline{v}_6\overline{v}_2\overline{v}_5\overline{v}_3\overline{v}_1v_1,\ \text{and}\quad v_1v_2v_3v_4v_5v_6\overline{v}_6\overline{v}_3\overline{v}_5\overline{v}_2\overline{v}_4\overline{v}_1v_1$$

are cycles of $T\overline{T}$ of lengths 10 to 12 such that the cycle of length 11 is good and the cycle of length 12 is very good, respectively. This completes the base case of the induction.

Now let $n \geq 7$.

We consider different cases.

Case 1: *Some vertex v of T is adjacent to at least 3 leaves of T.*

Let W denote the set of leaves of T that are adjacent to v and let $w \in W$. The choice of F implies $W \subseteq U$ and $v \in N$. Let $T^{(0)} = T - w$.

If either $|W| \geq 4$ or $|W| = 3$ and U contains an element of $N_T(v) \setminus W$, then let $F^{(0)} = F - w$ and let $U^{(0)} = U \setminus \{w\}$; otherwise let $F^{(0)} = F - (\{v\} \cup W)$ and let $U^{(0)} = U \setminus W$. In both cases $F^{(0)}$ is a special subforest of $T^{(0)}$ generated by $U^{(0)}$, which implies $f(T^{(0)}) \geq f(T) - 1$.

Now let $\tilde{F}^{(0)}$ be the special subforest of $T^{(0)}$ of maximum order. Let $\tilde{F}^{(0)}$ be generated by $\tilde{U}^{(0)}$ and let $\tilde{N}^{(0)} = N_{T^{(0)}}(\tilde{U}^{(0)})$. Since v has neighbors of degree 1 in $T^{(0)}$, we have $v \not\in \tilde{U}^{(0)}$. If $v \in \tilde{N}^{(0)}$, then $T[V(\tilde{F}^{(0)}) \cup \{w\}]$ is a special subforest of T, which implies $f(T) \geq f(T^{(0)}) + 1$. If $v \not\in \tilde{N}^{(0)}$, then $|W| = 3$, $W \cap V(\tilde{F}^{(0)}) = \emptyset$, and $\tilde{U}^{(0)}$ contains no element of $N_T(v) \setminus W$. Now $T[V(\tilde{F}^{(0)}) \cup (\{v\} \cup W)]$ is a special subforest of T, which also implies $f(T) \geq f(T^{(0)}) + 1$.

Altogether it follows that $f(T) = f(T^{(0)}) + 1$ and hence $F^{(0)}$ as defined above is the special subforest of $T^{(0)}$ of maximum order.

Since $T^{(0)}$ has order at least 6 and diameter at least 4, we obtain, by induction, the existence of two cycles $C^{(0)-}$ and $C^{(0)}$ of $T^{(0)}\overline{T^{(0)}}$ such that $(T^{(0)}, F^{(0)}, U^{(0)}, C^{(0)-}, C^{(0)})$ is good. Since $\overline{w}$ is adjacent to all vertices in $\overline{V}(T^{(0)}) \setminus \{\overline{v}\}$ and $C^{(0)}$ is very good, $C^{(0)}$ contains an edge $\overline{v}_a\overline{v}_b$ such that $\overline{w}$ is adjacent to $\overline{v}_a$ and $\overline{v}_b$. Hence replacing $\overline{v}_a\overline{v}_b$ in $C^{(0)}$ with $\overline{v}_a\overline{w}\overline{v}_b$ results in a very good cycle C of $T\overline{T}$ such that $(T, F, U, C^{(0)}, C)$ is good, which completes the proof in this case.

A path $P = v_0 \cdots v_r$ of length at least 3 in T is *maximal*, if v_1 and v_{r-1} have at most one neighbor in T that is not a leaf. Note that v_0 and v_r are necessarily leaves of T.

Case 2: *There is no maximal path $P = v_0 \cdots v_r$ in T such that the tree*

$$T - (\{v_1\} \cup (N_T(v_1) \setminus \{v_2\}))$$

has order at least 6 *and diameter at least* 4*.*

By Case 1 and the assumption of Case 2, the tree T is one of the twelve trees shown in Figure 3.

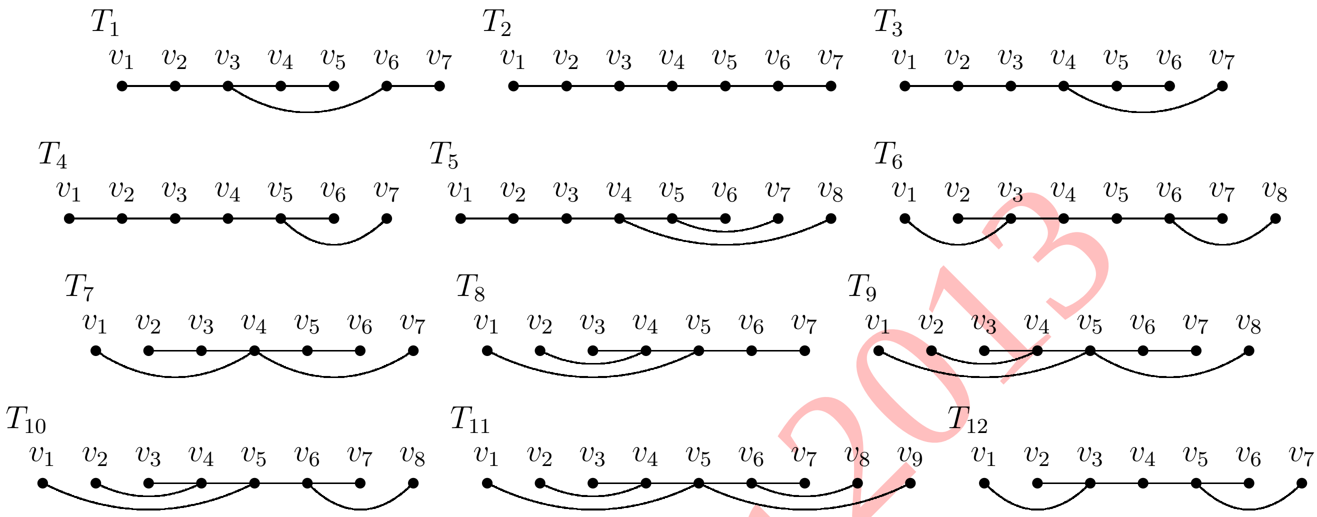

Figure 3: Twelve trees $T_1, \ldots, T_{12}$.

Note that each T_i has a leaf u such that $T_i - u$ has a pancyclic complementary prism. Therefore, since none of the trees $T_1, \ldots, T_{11}$ has a non-empty special subforest, it suffices, to exhibit for every $i \in [11]$, a good cycle C_i^- of $T_i\overline{T}_i$ of length $2|V(T_i)| - 1$ and a very good cycle C_i of $T_i\overline{T}_i$ of length $2|V(T_i)|$ such that $(T_i, \emptyset, \emptyset, C_i^-, C_i)$ is good. Similarly, since T_{12} is its own special subforest of maximum order generated by $\{v_1, v_2, v_4, v_6, v_7\}$, we have $f(T_{12}) = 2|V(T_{12})| - \mathrm{def}(T_{12}) = 13$ and it suffices, to exhibit a good cycle C_{12}^- of $T_{12}\overline{T}_{12}$ of length 12 and a very good cycle C_{12} of $T_{12}\overline{T}_{12}$ of length 13 such that $(T_{12}, T_{12}, \{v_1, v_2, v_4, v_6, v_7\}, C_{12}^-, C_{12})$ is good. Such cycles are given in Table 1.

i	C_i^-	C_i
1	$v_1v_2\overline{v}_2\overline{v}_7v_7v_6\overline{v}_6\overline{v}_4v_4v_5\overline{v}_5\overline{v}_3\overline{v}_1v_1$	$v_1v_2\overline{v}_2\overline{v}_6\overline{v}_4v_4v_5\overline{v}_5\overline{v}_3v_3v_6v_7\overline{v}_7\overline{v}_1v_1$
2	$v_1v_2v_3v_4v_5v_6v_7\overline{v}_7\overline{v}_3\overline{v}_6\overline{v}_2\overline{v}_5\overline{v}_1v_1$	$v_1v_2v_3v_4v_5v_6v_7\overline{v}_7\overline{v}_5\overline{v}_3\overline{v}_6\overline{v}_2\overline{v}_4\overline{v}_1v_1$
3	$v_1v_2v_3\overline{v}_3\overline{v}_5v_5v_6\overline{v}_6\overline{v}_4v_4v_7\overline{v}_7\overline{v}_1v_1$	$v_1v_2v_3\overline{v}_3\overline{v}_5v_5v_6\overline{v}_6\overline{v}_2\overline{v}_4v_4v_7\overline{v}_7\overline{v}_1v_1$
4	$v_1v_2\overline{v}_2\overline{v}_6v_6v_5v_7\overline{v}_7\overline{v}_4v_4v_3\overline{v}_3\overline{v}_1v_1$	$v_1v_2\overline{v}_2\overline{v}_6v_6v_5v_7\overline{v}_7\overline{v}_4v_4v_3\overline{v}_3\overline{v}_5\overline{v}_1v_1$
5	$v_1v_2\overline{v}_2\overline{v}_4\overline{v}_6v_6v_5v_7\overline{v}_7\overline{v}_8v_8v_4v_3\overline{v}_3\overline{v}_1v_1$	$v_1v_2\overline{v}_2\overline{v}_4\overline{v}_6v_6v_5v_7\overline{v}_7\overline{v}_8v_8v_4v_3\overline{v}_3\overline{v}_5\overline{v}_1v_1$
6	$v_1v_3v_2\overline{v}_2\overline{v}_4v_4v_5\overline{v}_5\overline{v}_3\overline{v}_7v_7v_6v_8\overline{v}_8\overline{v}_1v_1$	$v_1v_3v_2\overline{v}_2\overline{v}_6\overline{v}_4v_4v_5\overline{v}_5\overline{v}_3\overline{v}_7v_7v_6v_8\overline{v}_8\overline{v}_1v_1$
7	$v_1v_4v_7\overline{v}_7\overline{v}_3v_3v_2\overline{v}_2\overline{v}_6v_6v_5\overline{v}_5\overline{v}_1v_1$	$v_1v_4v_7\overline{v}_7\overline{v}_3v_3v_2\overline{v}_2\overline{v}_4\overline{v}_6v_6v_5\overline{v}_5\overline{v}_1v_1$
8	$v_1v_5\overline{v}_5\overline{v}_2v_2v_4v_3\overline{v}_3\overline{v}_7v_7v_6\overline{v}_6\overline{v}_1v_1$	$v_1v_5\overline{v}_5\overline{v}_2v_2v_4v_3\overline{v}_3\overline{v}_7v_7v_6\overline{v}_6\overline{v}_4\overline{v}_1v_1$
9	$v_1v_5v_8\overline{v}_8\overline{v}_2v_2v_4v_3\overline{v}_3\overline{v}_5\overline{v}_7v_7v_6\overline{v}_6\overline{v}_1v_1$	$v_1v_5v_8\overline{v}_8\overline{v}_2v_2v_4v_3\overline{v}_3\overline{v}_5\overline{v}_7v_7v_6\overline{v}_6\overline{v}_4\overline{v}_1v_1$
10	$v_1v_5\overline{v}_5\overline{v}_8v_8v_6v_7\overline{v}_7\overline{v}_3v_3v_4v_2\overline{v}_2\overline{v}_6\overline{v}_1v_1$	$v_1v_5\overline{v}_5\overline{v}_8v_8v_6v_7\overline{v}_7\overline{v}_3v_3v_4v_2\overline{v}_2\overline{v}_6\overline{v}_4\overline{v}_1v_1$
11	$v_1v_5v_9\overline{v}_9\overline{v}_4\overline{v}_7v_7v_6v_8\overline{v}_8\overline{v}_2v_2v_4v_3\overline{v}_3\overline{v}_6\overline{v}_1v_1$	$v_1v_5v_9\overline{v}_9\overline{v}_4\overline{v}_7v_7v_6v_8\overline{v}_8\overline{v}_5\overline{v}_2v_2v_4v_3\overline{v}_3\overline{v}_6\overline{v}_1v_1$
12	$v_1v_3v_2\overline{v}_2\overline{v}_5\overline{v}_3\overline{v}_6v_6v_5v_7\overline{v}_7\overline{v}_1v_1$	$v_1v_3v_2\overline{v}_2\overline{v}_5\overline{v}_3\overline{v}_6v_6v_5v_7\overline{v}_7\overline{v}_4\overline{v}_1v_1$

Table 1: Good and very good cycles of the trees in Figure 3.

In view of Case 2, we may assume now that $P = v_0 \cdots v_r$ is a maximal path in T such that $T - (\{v_1\} \cup (N_T(v_1) \setminus \{v_2\}))$ has order at least 6 and diameter at least 4. By Case 1, we may assume that the degree of v_1 is 2 or 3.

Below we consider further subtrees $T^{(i)}$ of T. The maximum special subforest of $T^{(i)}$ will be denoted by $F^{(i)}$ and the set generating $F^{(i)}$ will be denoted by $U^{(i)}$. Furthermore, $N^{(i)}$ will denote $N_{T^{(i)}}(U^{(i)})$.

Case 3: *The degree of* v_1 *in* T *is* 2.

Let $T^{(1)} = T - \{v_0, v_1\}$ and $T^{(2)} = T - v_0$.

First we assume that $v_2 \not\in U^{(1)}$. This implies that $F^{(1)} = F$ and hence $f(T) = f(T^{(1)}) - 4$. If D is a good cycle of $T^{(1)}\overline{T^{(1)}}$ of length ℓ that uses the two independent edges $\overline{v}_a\overline{v}_b$ and $\overline{v}_c\overline{v}_d$, then we may assume that $\overline{v}_1$ is adjacent to $\overline{v}_a$ and $\overline{v}_b$. Replacing $\overline{v}_a\overline{v}_b$ with $\overline{v}_a\overline{v}_1\overline{v}_b$ or with $\overline{v}_a\overline{v}_1v_1v_0\overline{v}_0\overline{v}_b$ and/or replacing $\overline{v}_c\overline{v}_d$ with $\overline{v}_c\overline{v}_0\overline{v}_d$, allows to construct good cycles of $T\overline{T}$ of lengths $\ell+1$, $\ell+2$, and $\ell+4$. Note that, if D is very good, then also the constructed cycles are very good. By induction, there are cycles $C^{(1)-}$ and $C^{(1)}$ in $T^{(1)}\overline{T^{(1)}}$ such that $(T^{(1)}, F^{(1)}, U^{(1)}, C^{(1)-}, C^{(1)})$ is good. Applying the extensions described above to $C^{(1)-}$ and $C^{(1)}$ implies the existence of cycles C^- and C in $T\overline{T}$ such that (T, F, U, C^-, C) is good.

Next we assume that $v_2 \in U^{(1)}$. For a contradiction, we assume that $v_1 \in U^{(2)}$. This implies that $v_2 \in U^{(1)} \cap N^{(2)}$. By definition, every vertex in $U^{(1)} \cap N^{(2)}$ has at least 2 neighbors in $T^{(1)}$ that belong to $U^{(2)} \cap N^{(1)}$ and every vertex in $U^{(2)} \cap N^{(1)}$ has at least 2 neighbors in $T^{(1)}$ that belong to $U^{(1)} \cap N^{(2)}$, that is, v_2 lies in a subgraph of $T^{(1)}$ of minimum degree at least 2, which is impossible because T is a tree. Hence $v_1 \not\in U^{(2)}$. This implies that $F^{(2)} = F$ and hence $f(T) = f(T^{(2)}) - 2$. By induction, there are cycles $C^{(2)-}$ and $C^{(2)}$ in $T^{(2)}\overline{T^{(2)}}$ such that $(T^{(2)}, F^{(2)}, U^{(2)}, C^{(2)-}, C^{(2)})$ is good. Since v_1 has degree 2 in $T^{(2)}\overline{T^{(2)}}$, the cycle $C^{(2)}$ contains the path $\overline{v}_1 v_1 v_2$. Since $C^{(2)}$ is very good, it contains three independent edges, say $\overline{v}_a\overline{v}_b$, $\overline{v}_c\overline{v}_d$, and $\overline{v}_e\overline{v}_f$, of $\overline{T^{(2)}}$. Clearly, we may assume that $\overline{v}_1$ is not incident with $\overline{v}_a\overline{v}_b$ or $\overline{v}_c\overline{v}_d$. Since $\overline{v}_1$ is adjacent to all vertices in $\overline{V}(T^{(2)}) \setminus \{\overline{v}_2\}$, we may further assume that $\overline{v}_1$ is adjacent to $\overline{v}_a$ and $\overline{v}_b$. Replacing the edge $\overline{v}_a\overline{v}_b$ by the path $\overline{v}_a\overline{v}_0\overline{v}_b$ results in a (very) good cycle C^- of $T\overline{T}$ of length $f(T) - 1$. Replacing the two edges $\overline{v}_a\overline{v}_b$ and $v_1\overline{v}_1$ of $C^{(2)}$ either by the paths $\overline{v}_a\overline{v}_1$ and $\overline{v}_b\overline{v}_0 v_0 v_1$ or by the paths $\overline{v}_b\overline{v}_1$ and $\overline{v}_a\overline{v}_0 v_0 v_1$ results in a very good cycle C of $T\overline{T}$ of length $f(T)$, that is, (T, F, U, C^-, C) is good.

Case 4: *The degree of v_1 in T is 3.*

Let w_0 denote the neighbor of v_1 distinct from v_0 and v_2. Let $T^{(3)} = T - \{v_0, v_1, w_0\}$.

If $v_2 \in U$, then $v_1 \in N$ and $v_0, w_0 \in U$. This implies that $F - \{v_0, v_1, w_0\}$ is a special induced forest of $T^{(3)}$. Lemma 5 implies that $V(F) \setminus \{v_0, v_1, w_0\}$ is a subset of $V(F^{(3)})$ and hence $U \setminus \{v_0, v_1, w_0\}$ is a subset of $U^{(3)}$, which implies $v_2 \in U^{(3)}$. This implies that $T[V(F^{(3)}) \cup \{v_0, v_1, w_0\}]$ is a special induced subforest of T. Altogether, we obtain $f(T) = f(T^{(3)}) - 6$.

If $v_2 \not\in U$, then $v_0, v_1, w_0 \not\in V(F)$ and F is a special induced forest of $T^{(3)}$. If $v_2 \in U^{(3)}$, then, by Lemma 5, $U \cup U^{(3)} \cup \{v_0, w_0\}$ generates a special subforest of T whose order is larger than the order of F, which is a contradiction. Hence $v_2 \not\in U^{(3)}$, which implies that $F^{(3)}$ is a special induced forest of T. Again we obtain $f(T) = f(T^{(3)}) - 6$.

By induction, there are two cycles $C^{(3)-}$ and $C^{(3)}$ such that $(T^{(3)}, F^{(3)}, U^{(3)}, C^{(3)-}, C^{(3)})$ is good. Let $\overline{v}_a\overline{v}_b$, $\overline{v}_c\overline{v}_d$, and $\overline{v}_e\overline{v}_f$ denote the three independent edges used by the very good cycle $C^{(3)}$. Clearly, we may assume that $\overline{v}_1$ is adjacent to $\overline{v}_a$ and $\overline{v}_b$. Replacing $\overline{v}_a\overline{v}_b$ with $\overline{v}_a\overline{v}_1\overline{v}_b$ and/or $\overline{v}_c\overline{v}_d$ with $\overline{v}_c\overline{v}_0\overline{v}_d$ and/or $\overline{v}_e\overline{v}_f$ with $\overline{v}_e\overline{w}_0\overline{v}_f$ in $C^{(3)}$ results in cycles of $T\overline{T}$ of lengths $f(T)-5$, $f(T)-4$, and $f(T)-3$. Replacing $\overline{v}_a\overline{v}_b$ with $\overline{v}_a\overline{v}_1v_1v_0\overline{v}_0\overline{v}_b$ in $C^{(3)}$ results in a cycle of $T\overline{T}$ of length $f(T)-2$. Replacing $\overline{v}_a\overline{v}_b$ with $\overline{v}_a\overline{w}_0w_0v_1v_0\overline{v}_0\overline{v}_b$ in $C^{(3)}$ results in a good cycle C^- of $T\overline{T}$ of length $f(T)-1$. Finally, replacing $\overline{v}_a\overline{v}_b$ with $\overline{v}_a\overline{v}_1\overline{v}_b$ and $\overline{v}_c\overline{v}_d$ with $\overline{v}_c\overline{w}_0w_0v_1v_0\overline{v}_0\overline{v}_d$ in $C^{(3)}$ results in a very good cycle C of $T\overline{T}$ of length $f(T)$. Altogether (T, F, U, C^-, C) is good, which completes the proof. $\square$

Backwards engineering the proof of Theorem 2 allows to efficiently construct cycles of all possible lengths in the complementary prism of a given tree.

3 Graphs of Bounded Maximum Degree

We now consider the following decision problem.

Hamiltonicity of Complementary Prisms

Input: A graph G.

Question: Is $G\overline{G}$ Hamiltonian?

As pointed out by Fleischner [6], the analogous problem for ordinary prisms is hard; more specifically, it is NP-complete to decide for a given graph whether its prism is Hamiltonian. The situation for complementary prisms could be different. Intuitively speaking, it seems plausible that Hamiltonian cycles of $G\overline{G}$ can be constructed by linking suitable paths in one of G or $\overline{G}$ with the help of the other one relying on the fact that one of these two graphs is necessarily dense. Our next results show that this intuition is valid when we force G to be sparse by imposing an upper bound on its maximum degree.

Lemma 10. *Let Δ be an integer. For every graph G of order $n \geq 3\Delta(3\Delta+2)+1$ and minimum degree at least $n-1-\Delta$ and for every matching M of G, there is a Hamiltonian cycle C of G with $M \subseteq E(C)$.*

Proof. Let G be a graph of order $n \geq 3\Delta(3\Delta+2)+1$ and minimum degree at least $n-1-\Delta$, that is every vertex u of G is non-adjacent to at most Δ vertices in $V(G) \setminus \{u\}$. Let $M = \{x_1y_1, \dots, x_ry_r\}$ be a matching of G. Let $V(M) = \{x_1, y_1, \dots, x_r, y_r\}$ and $Z = V(G) \setminus V(M) = \{z_1, \dots, z_s\}$.

Recall that a graph of order p and maximum degree q has an independent set of order at least $p/(q+1)$. It follows that there is a subset Z' of at least $s/(\Delta+1)$ vertices in Z such that Z' induces a clique.

Let H be the auxiliary graph on the vertex set $\{v_1, \dots, v_r\}$ that contains an edge v_iv_j if and only if $G[\{x_i, y_i, x_j, y_j\}] = K_4$. By construction, H has minimum degree at least $r-1-2\Delta$ and thus, it has a clique of order at least $r/(2\Delta+1)$. It follows that there is a set M' of at least $2r/(2\Delta+1)$ edges in M such that $V(M')$ induces a clique.

Hence, if $n \geq 3\Delta(3\Delta+2)+1$, the graph G contains a cycle C of length ℓ with $\ell > 3\Delta$ such that for every vertex u on C that belongs to $V(M)$, the edge in M incident with u belongs to C.

If $\ell < n$, then there is either an edge $xy \in M$ with $x, y \notin V(C)$ or a vertex $z \in Z$ with $z \notin V(C)$. In the first case, let N denote the set of neighbors of x on C. Clearly, $|N| \geq \ell - \Delta$. Since M is a matching, every vertex u in N has a neighbor denoted by $p(u)$ such that the edge $up(u)$ belongs to C but not to M. There is a subset N' of N with $|N'| \geq |N|/2$ such that for distinct vertices u and v in N', the vertices $p(u)$ and $p(v)$ are distinct. Since $(\ell-\Delta)/2 > \Delta$, there is some u in N' such that y is adjacent to $p(u)$. Now the cycle C can be extended by replacing the edge $up(u)$ with the path $uxyp(u)$. In the second case that there is a vertex $z \in Z$ with $z \notin V(C)$, a similar argument shows that the cycle can be extended. □

Theorem 11. *The decision problem* HAMILTONICITY OF COMPLEMENTARY PRISMS *restricted to instance graphs of maximum degree Δ and order $n \geq 3\Delta(3\Delta+2)+1$ can be solved in polynomial time.*

Proof. Let G be a graph of maximum degree Δ and order $n \geq 3\Delta(3\Delta+2)+1$. If C is a Hamiltonian cycle of $G\overline{G}$, then the intersection $C \cap G$ is a factor of G in which all components are paths of length at least 1. Conversely, let G have a factor H in which all components are paths of length at least 1. For every path P_i in H between some vertices x_i and y_i, we add to $\overline{G}$ the edge $\overline{x}_i\overline{y}_i$ to form the graph $\overline{G}^+$. Clearly, the added edges form a matching M of $\overline{G}^+$ and $\overline{G}^+$ has minimum degree at least $n-1-\Delta$. Therefore, if $n \geq 3\Delta(3\Delta+2)+1$, then Lemma 10 implies the existence of a Hamiltonian cycle C of $\overline{G}^+$ with $M \subseteq E(C)$. Replacing each edge $\overline{x}_i\overline{y}_i$ in C with the path $\overline{x}_iP_i\overline{y}_i$ results in a Hamiltonian cycle of $G\overline{G}$. Altogether this implies that for $n \geq 3\Delta(3\Delta+2)+1$, $G\overline{G}$ is Hamiltonian if and only if G has a factor in which all components are paths of length at least 1. Note that such a factor exists if and only if G has a factor F in which every vertex has degree 1 or 2. By observations of Lovász [13] and Tutte [16, 17], such a factor exists if and only if some auxiliary graph whose order is polynomially bounded in n has a perfect matching. Altogether, if n is sufficiently large, then Hamiltonicity of Complementary Prisms can be solved using Edmonds' maximum matching algorithm [5], and if n is not sufficiently large, then it can be solved by brute force. $\square$

One of the most interesting open problems related to complementary prisms concerns the complexity of Hamiltonicity of Complementary Prisms.

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