The Erdős-Pósa Property for Long Circuits

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Abstract

For an integer ℓ at least 3, we prove that if G is a graph containing no two vertex-disjoint circuits of length at least ℓ, then there is a set X of at most $\frac{5}{3}\ell + \frac{29}{2}$ vertices that intersects all circuits of length at least ℓ. Our result improves the bound $2\ell + 3$ due to Birmelé, Bondy, and Reed (The Erdős-Pósa property for long circuits, Combinatorica 27 (2007), 135-145) who conjecture that ℓ vertices always suffice.

Keywords: Erdős-Pósa property; circuit; packing; covering

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1 Introduction

A family $\mathcal{F}$ of graphs is said to have the *Erdős-Pósa property* if there is a function $f_{\mathcal{F}} : \mathbb{N} \to \mathbb{N}$ such that for every graph G and every $k \in \mathbb{N}$, either G contains k vertex-disjoint subgraphs that belong to $\mathcal{F}$ or there is a set X of at most $f_{\mathcal{F}}(k)$ vertices of G such that $G - X$ has no subgraph that belongs to $\mathcal{F}$. The origin of this notion is [3] where Erdős and Pósa prove that the family of all circuits has this property.

Let ℓ be an integer at least 3. Let $\mathcal{F}_\ell$ denote the family of circuits of length at least ℓ. In [2] Birmelé, Bondy, and Reed show that $\mathcal{F}_\ell$ has the Erdős-Pósa property with

$$f_{\mathcal{F}_\ell}(k) \leq 13\ell(k-1)(k-2) + (2\ell+3)(k-1), \tag{1}$$

which improves an earlier doubly exponential bound on $f_{\mathcal{F}_\ell}(k)$ obtained by Thomassen [5]. The main contribution of Birmelé, Bondy, and Reed [2] is to prove (1) for $k = 2$, that is, to show

$$f_{\mathcal{F}_\ell}(2) \leq 2\ell + 3, \tag{2}$$

For $k \geq 3$, an inductive argument allows to deduce (1) from (2).

Birmelé, Bondy, and Reed [2] conjecture that

$$f_{\mathcal{F}_\ell}(2) \leq \ell, \tag{3}$$

that is, for every graph G containing no two vertex-disjoint circuits of length at least ℓ, there is a set X of at most ℓ vertices such that $G - X$ has no circuit of length at least ℓ. In view of the complete graph of order $2\ell - 1$, (3) would be best possible. For $\ell = 3$, (3) was shown by Lovász [4] and for $\ell \in \{4, 5\}$, (3) was shown by Birmelé [1].

Our contribution in the present paper is the following result.

Theorem 1 *Let ℓ be an integer at least 3. Let G be a graph containing no two vertex-disjoint circuits of length at least ℓ.*

There is a set X of at most $\frac{5}{3}\ell + \frac{29}{2}$ vertices that intersects all circuits of length at least ℓ, that is,

$$f_{\mathcal{F}_\ell}(2) \leq \frac{5}{3}\ell + \frac{29}{2}.$$

While Theorem 1 is a nice improvement of (2), for $k \geq 3$, the above-mentioned inductive argument still leads to an estimate of the form $f_{\mathcal{F}_\ell}(k) = O(\ell k^2)$.

The rest of this paper is devoted to the proof of Theorem 1.

2 Proof of Theorem 1

With respect to notation and terminology we follow [2] and recall some specific notions. All graphs are finite, simple, and undirected. We abbreviate *vertex-disjoint* as *disjoint*. If A and B are sets of vertices of a graph G, then an (A, B)*-path* is a path P in G between a vertex in A and a vertex in B such that no internal vertex of P belongs to $A \cup B$. If P is a path and x and y are vertices of P, then $P[x, y]$ denotes the subpath of P between x and y. Similarly, is C is a circuit endowed with an orientation and x and y are vertices of C, then $C[x, y]$ denotes the segment of C from x to y following the orientation of C. In all figures of circuits the orientations will be counterclockwise.

We fix an integer ℓ at least 3 and call a circuit of length at least ℓ *long*.

If C is a circuit and P and P' are disjoint $(V(C), V(C))$-paths such that P is between u and v and P' is between u' and v', then

- P and P' are called *parallel (with respect to C)* if u, u', v', v appear in the given cyclic order on C and
- P and P' are called *crossing (with respect to C)* if u, u', v, v' appear in the given cyclic order on C.

See Figure 1.

In the proof of Theorem 1 below we consider three cases according to the length L of a shortest long circuit. If L is less than $3\ell/2$, the result is trivial. For L between $3\ell/2$ and 2ℓ the following lemma implies the desired bound. Finally, for L larger than 2ℓ, Lemma 3 implies the desired bound.

Lemma 2 *Let G be a graph containing no two disjoint long circuits.*

If the shortest long circuit of G has length L with $L \geq 3\left(\left\lceil \frac{1}{2}\ell \right\rceil - 2\right)$, then there is a set X of at most $\frac{1}{3}L + \ell + \frac{14}{3}$ vertices that intersects all long circuits.

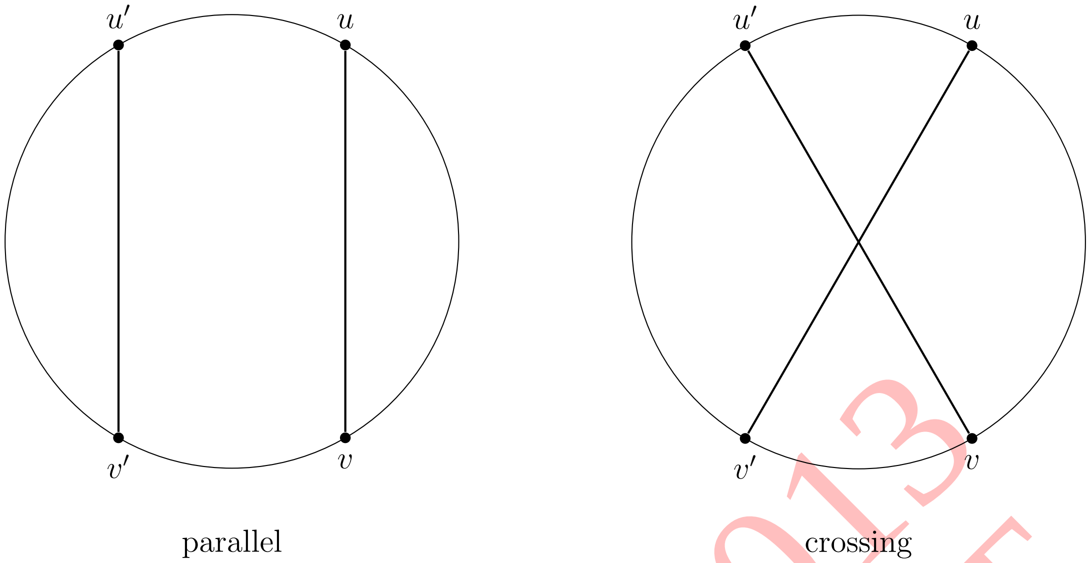

Figure 1: Parallel and crossing pairs of paths.

Proof: Let C be a shortest long circuit of G. We endow C with an orientation. We decompose C into 6 cyclically consecutive and internally disjoint segments $C_1, \ldots, C_6$ such that C_1, C_3, and C_5 have length $\lceil \frac{1}{2}\ell \rceil - 2$ and C_2, C_4, and C_6 have lengths between $\lceil \frac{1}{3}L - (\lceil \frac{1}{2}\ell \rceil - 2) \rceil$ and $\lceil \frac{1}{3}L - (\lceil \frac{1}{2}\ell \rceil - 2) \rceil + 1$, that is, the six segments cover all of C and C_i and C_{i+1} overlap in exactly one vertex for every $i \in [6]$ where we identify indices modulo 6.

Let $X_1 = V(C_1) \cup V(C_3) \cup V(C_5)$. See the left part of Figure 2.
Let $i \in [6]$ be even. Let $\mathcal{P}_i$ denote the set of $(V(C_i), V(C_{i+2}))$-path in $G - (X_1 \cup V(C_{i+4}))$. The choice of C implies that every path P in $\mathcal{P}_i$ has length at least $\frac{1}{2}\ell$; otherwise P together with a segment of C avoiding $V(C_{i+1})$ forms a long circuit that is shorter than C. See the right part of Figure 2. This implies that for every path P in $\mathcal{P}_i$, P together with a segment of C containing $V(C_{i+1})$ forms a long circuit.

Let $\mathcal{P} = \mathcal{P}_2 \cup \mathcal{P}_4 \cup \mathcal{P}_6$. Since G has no two disjoint long circuits, it follows that $\mathcal{P}$ contains no two disjoint parallel paths and no four disjoint crossing paths. See Figure 3.
Let X_2 be a smallest set of vertices separating $V(C_2)$ and $V(C_4) \cup V(C_6)$ in $G - X_1$. Let X_3 be a smallest set of vertices separating $V(C_4)$ and $V(C_6)$ in $G - (X_1 \cup X_2)$. By the above observations and Menger's theorem, $|X_2| \leq 3$ and $|X_3| \leq 3$.

There is some even $j \in [6]$ such that in $G - (X_1 \cup X_2 \cup X_3)$, all long circuits intersect C only in $V(C_j)$; otherwise there is a $(V(C_i), V(C_{i+2}))$-path in $G - (X_1 \cup X_2 \cup X_3)$ for some even $i \in [6]$. This implies that $X_1 \cup X_2 \cup X_3 \cup V(C_j)$ intersects all long circuits of G. Since

$$\begin{aligned} |X_1 \cup X_2 \cup X_3 \cup V(C_j)| &\leq 3\left(\left\lceil \frac{1}{2}\ell \right\rceil - 2\right) + 3 + 3 + \left\lceil \frac{1}{3}L - \left(\left\lceil \frac{1}{2}\ell \right\rceil - 2\right) \right\rceil + 1 \\ &\leq \frac{1}{3}L + \ell + \frac{14}{3} \end{aligned}$$

we obtain the desired result. $\square$

Lemma 3 *Let G be a graph containing no two disjoint long circuits.*

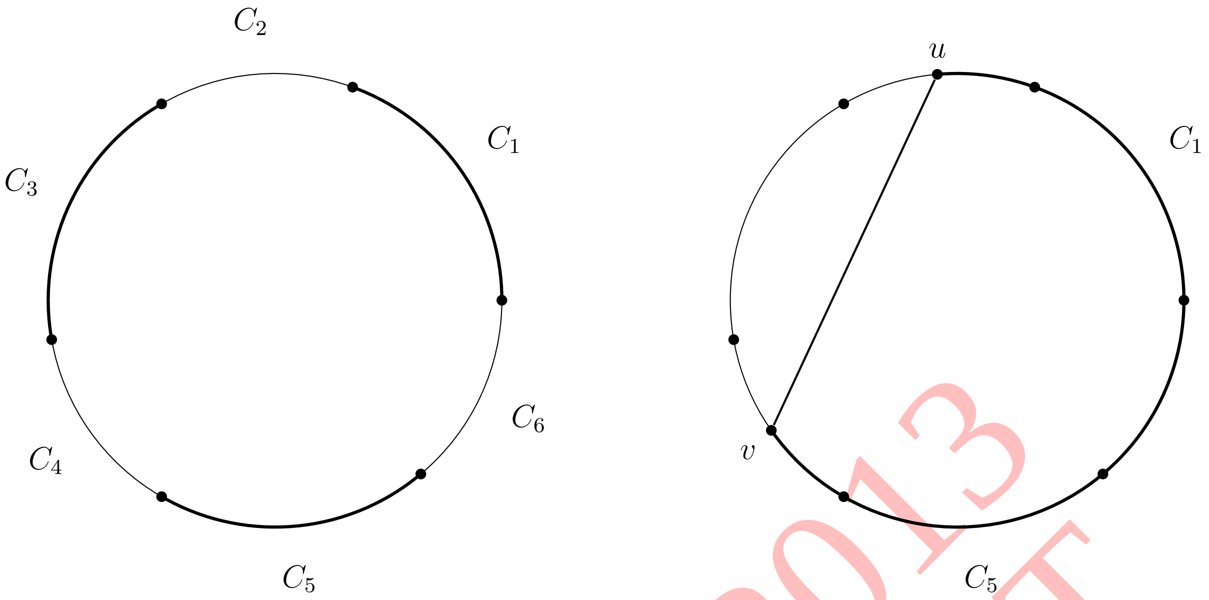

Figure 2: On the left the six segments of C and the set X in bold. On the right a long circuit formed by a $(V(C_2), V(C_4))$-path between u and v in $G - (X_1 \cup V(C_6))$.

If the shortest long circuit of G has length at least $2\ell - 3$, then there is a set X of at most $\frac{3}{2}\ell + \frac{29}{2}$ vertices that intersects all long circuits.

Proof: Let C be shortest long circuit of G. Let L denote the length of C. We endow C with an orientation.

As in [2], a path between two vertices x and y of C that is internally disjoint from C is called *long*, if the segments $C[x, y]$ and $C[y, x]$ both have length at least $\frac{1}{2}\ell$.

Claim A *Every long path has length at least $\ell - 1$.*

Proof of Claim A: Let P be a long path between two vertices x and y of C. Let L_P denote the length of P. We may assume that $C[x, y]$ is at least as long as $C[y, x]$. Since $C[x, y]$ has length at least $\ell - 1$, the union of P and $C[x, y]$ is a long circuit. Since this circuit has length at least L, the length of P is at least the length of $C[y, x]$, that is, $L_P \geq \frac{1}{2}\ell$. Now it follows that the union of P with $C[y, x]$ is also a long circuit of length at most $\frac{L}{2} + L_P$. Since this is at least L, we obtain $L_P \geq \frac{L}{2}$, that is, $L_P \geq \ell - 1$. □

Choose a long circuit D of G distinct from C and a segment $C[x, y]$ of C such that $C[x, y]$ contains $V(C) \cap V(D)$ and has minimum possible length. Note that $x, y \in V(C) \cap V(D)$.

We consider two cases.

Case 1 $x \neq y$.

Let X_1 denote the set of $\lceil \frac{1}{2}\ell \rceil - 1$ vertices immediately preceeding x and let X_2 denote the set of $\lceil \frac{1}{2}\ell \rceil - 1$ vertices immediately following y. Let $A = V(C) \setminus (X_1 \cup X_2 \cup V(C[x, y]))$ and $B = V(C[x, y])$. See Figure 4.

In $G - (X_1 \cup X_2)$, there are no two disjoint parallel (A, B)-paths and no four disjoint crossing (A, B)-paths; otherwise there would be two disjoint long circuits. Hence, by Menger's theorem, there is a set X_3 of at most 3 vertices separating A and B in $G - (X_1 \cup X_2)$.

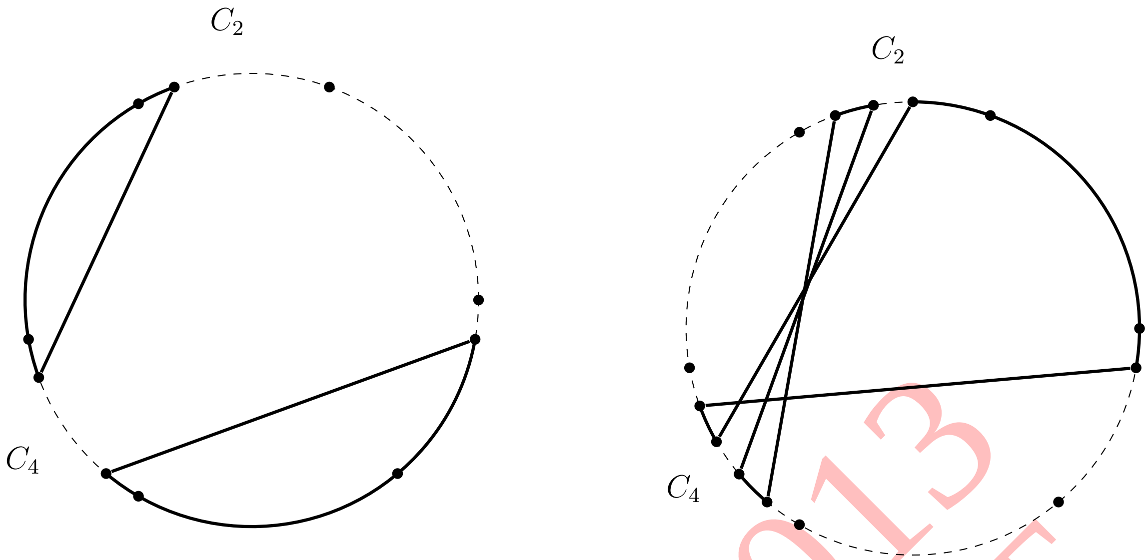

Figure 3: Two disjoint long circuits formed by two disjoint parallel paths in $\mathcal{P}$ or by four disjoint crossing paths in $\mathcal{P}$.

The circuit D uniquely decomposes into a set $\mathcal{P}$ of at least two (B, B)-paths of length at least 1. Note that $\mathcal{P}$ contains either no, or one, or two paths between x and y, depending on the intersection of C and D.

Claim B *If $\mathcal{P}$ contains a path P between x and y, then in $G - (X_1 \cup X_2 \cup X_3)$, there are at most $\lceil \frac{1}{2}\ell \rceil + 1$ disjoint $(A, V(P))$-paths.*

Proof of Claim B: For contradiction, we assume that there are at least $\lceil \frac{1}{2}\ell \rceil + 2$ such paths. Let $P_1, \ldots, P_k$ be an ordering of the paths according to their endpoints on $C[y, x]$. For $i \in [k]$, let P_i be between $x_i \in A$ and $y_i \in V(P)$. Let C_P denote the circuit $P \cup C[y, x]$. See Figure 4.

If two of these paths, say P_i and P_j, are parallel with respect to C_P, then $G - (X_1 \cup X_2)$ contains the two parallel (A, B)-paths $P_i \cup P[y_i, y]$ and $P_j \cup P[y_j, x]$, which is a contradiction. See the left part of Figure 4.

Hence all these paths are crossing respect to C_P. Now

$$C' = P_1 \cup P[y_1, x] \cup C_P[x_k, x] \cup P_k \cup P[y_k, y] \cup C[y, x_1]$$

is a long circuit containing X_1 and X_2. Furthermore, since $k - 2 \geq \frac{1}{2}\ell$,

$$P_2 \cup C_P[y_2, y_{k-1}] \cup P_{k-1} \cup C[x_2, x_{k-1}]$$

is a long circuit that is disjoint from C', which is a contradiction. See the right part of Figure 4. □

Claim C *If $\mathcal{P}$ contains two paths, say P and P', between x and y, then in $G - (X_1 \cup X_2 \cup X_3)$, there are at most $\lceil \frac{1}{2}\ell \rceil + 1$ disjoint $(A, V(D))$-paths.*

Proof of Claim C: Note that in this case, D decomposes into exactly two paths between x and y, that is, $V(D) \cap V(C) = \{x, y\}$ and $\mathcal{P} = \{P, P'\}$.

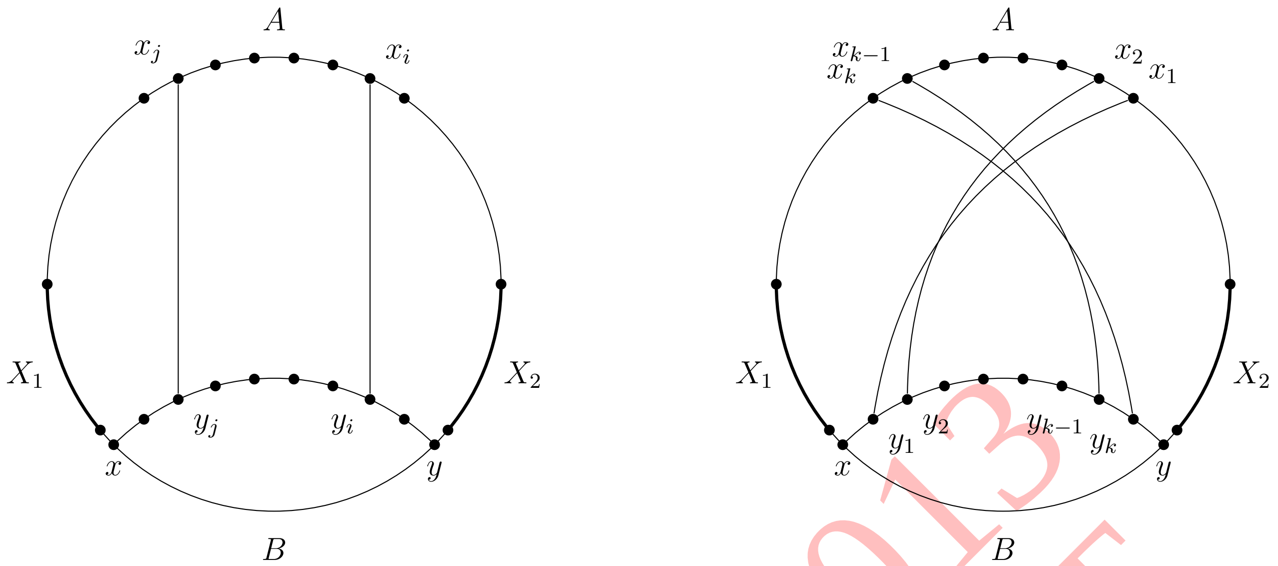

Figure 4: Two disjoint parallel $(A, V(P))$-paths P_i and P_j in $G - (X_1 \cup X_2 \cup X_3)$ and four disjoint crossing $(A, V(P))$-paths P_1, P_2, P_{k-1}, and P_k in $G - (X_1 \cup X_2 \cup X_3)$.

For contradiction, we assume that there are at least $\lceil \frac{1}{2}\ell \rceil + 2$ disjoint $(A, V(D))$-paths $G - (X_1 \cup X_2 \cup X_3)$. Claim B implies that there are two disjoint paths, one that is a $(A, V(P))$-path and one that is a $(A, V(P'))$-path. Since both paths avoid x and y, the graph $G - (X_1 \cup X_2)$ contains two parallel (A, B)-paths, which is a contradiction. $\Box$

Claim D *If P is a path in $\mathcal{P}$ that is not a path between x and y, then in $G - (X_1 \cup X_2 \cup X_3)$, there are at most 3 disjoint $(A, V(P))$-paths.*

Proof of Claim D: Let the path P be between x' and y' such that x, x', y', y appear in the given order on $C[x, y]$. See Figure 5. If the length of P is at least $\ell - 1$, then P together with $C[x', y']$ forms a long circuit D' such that $V(D') \cap V(C)$ is contained in a segment of C that is strictly smaller then $C[x, y]$, which contradicts the choice of D and $C[x, y]$. Hence the length of P is at most $\ell - 2$.

Let Q be a $(A, V(P))$-path in $G - (X_1 \cup X_2 \cup X_3)$ of length L_Q between $u \in A$ and $v \in V(P)$. See Figure 5. We may assume that $P[v, x']$ is at most as long as $P[v, y']$, that is, $P[v, x']$ has length at most $\frac{1}{2}\ell - 1$. $Q \cup P[v, x']$ is a long path of length at most $L_Q + \frac{1}{2}\ell - 1$. By Claim A, this length is at least $\ell - 1$, which implies $L_Q \geq \frac{1}{2}\ell$.

For contradiction, we assume now that there are four disjoint $(A, V(P))$-paths in $G - (X_1 \cup X_2 \cup X_3)$. By the previous observation, all these paths are of length at least $\frac{1}{2}\ell$. If two of these paths are parallel with respect to the circuit $P \cup C[y', x']$, then there are two disjoint long circuits, one containing X_1 and one containing X_2. If all four of these paths are crossing with respect to the circuit $P \cup C[y', x']$, then there are two disjoint long circuits avoiding X_1 and X_2; similarly as in the right part of Figure 3. These contradictions complete the proof. $\Box$

Claim E *If $P_1, \ldots, P_4$ are four distinct paths in $\mathcal{P}$ that are no paths between x and y, then in $G - (X_1 \cup X_2 \cup X_3)$, there are no four disjoint paths $Q_1, \ldots, Q_4$ such that Q_i is a $(A, V(P_i))$-path for $i \in [4]$.*

Proof of Claim E: For contradiction, we assume that such paths exist. Since none of the four

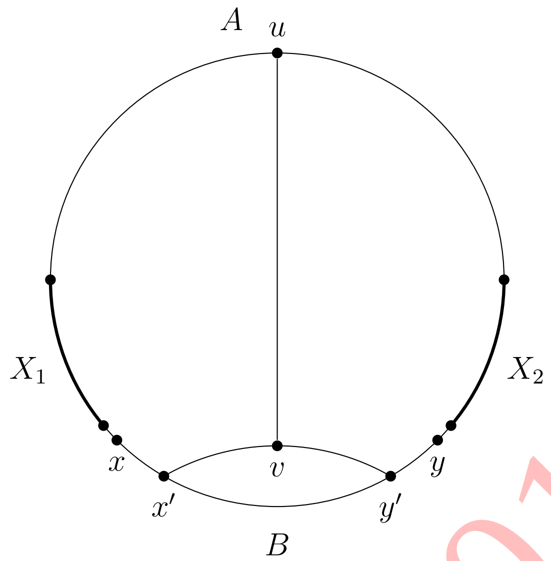

Figure 5: A $(A, V(P))$-path in $G - (X_1 \cup X_2 \cup X_3)$ between $u \in A$ and $v \in V(P)$.

paths $P_1, \ldots, P_4$ is between x and y, the union of all four paths is either a forest or equal to D. In the latter case, we may assume that P_1 and P_2 intersect in x.

In the first case we can select four disjoint (A, B)-paths $R_1, \ldots, R_4$ in $G - (X_1 \cup X_2)$ such that R_i is a path in $P_i \cup Q_i$ for $i \in [4]$. In the second case we can select two disjoint parallel (A, B)-paths R_1, R_2 in $G - (X_1 \cup X_2)$ such that R_i is a path in $P_i \cup Q_i$ for $i \in [2]$. As noted above, both cases lead to a contradiction. $\square$

Let V_1 denote the set of vertices r of D such that $\mathcal{P}$ contains a path between x and y that contains r and let V_2 denote the set of vertices s of D such that $\mathcal{P}$ contains a path not between x and y that contains s. Clearly, $V_1 \cup V_2 = V(D)$. By Claims B and C and Menger's theorem, there is a set X_4 of at most $\lceil \frac{1}{2}\ell \rceil + 1$ vertices separating A and V_1 in $G - (X_1 \cup X_2 \cup X_3)$. By Claims D and E and Menger's theorem, there is a set X_5 of at most 9 vertices separating A and V_2 in $G - (X_1 \cup X_2 \cup X_3)$.

Let $X = \{x, y\} \cup X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5$.

If $G - X$ contains a long circuit, say D', then D' intersects A. Since D and D' intersect, there is an $(A, V(D))$-path P in $G - X$. In view of X_3, P cannot end in B; in view of X_4, P cannot end in V_1; and, in view of X_5, P cannot end in V_2, which is a contradiction. Hence X intersects all long circuits. Since

$$\begin{aligned} |X| &\leq 2 + |X_1| + |X_2| + |X_3| + |X_4| + |X_5| \\ &\leq 2 + \left(\left\lceil \frac{1}{2}\ell \right\rceil - 1 \right) + \left(\left\lceil \frac{1}{2}\ell \right\rceil - 1 \right) + 3 + \left(\left\lceil \frac{1}{2}\ell \right\rceil + 1 \right) + 9 \\ &\leq \frac{3}{2}\ell + \frac{29}{2}, \end{aligned}$$

this completes the proof in the first case.

Case 2 $x = y$.

Clearly, we may assume that $G - \{x\}$ contains at least one long circuit. Since every long circuit in $G-\{x\}$ intersects D, there are $(V(C),V(D))$-paths in $G-\{x\}$. We choose a $(V(C),V(D))$-path P_0 in $G-\{x\}$ between a vertex $y \in V(C)$ and a vertex $z \in V(D)$ such that the distance in C between x and y is minimum. We may assume that $C[x,y]$ is a shortest path in C between x and y. See Figure 6.

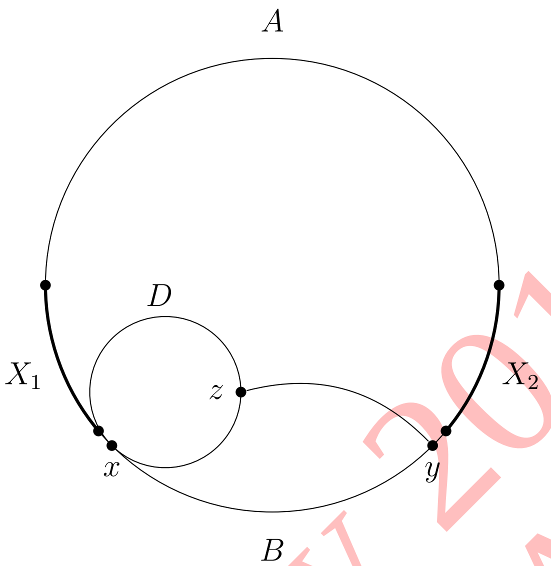

Figure 6: A $(V(C),V(D))$-paths in $G-\{x\}$ between $y \in V(C)$ and $z \in V(D)$.

We denote the two paths in D between x and z by P_1' and P_2'. Let $P_i = P_0 \cup P_i'$ for $i \in [2]$, that is, P_1 and P_2 are two paths between x and y that have the common segment P_0 and are internally disjoint from C.

Let X_1 denote the set of $\lceil \frac{1}{2}\ell \rceil - 1$ vertices immediately preceeding x and let X_2 denote the set of $\lceil \frac{1}{2}\ell \rceil - 1$ vertices immediately following y. Let $A = V(C) \setminus (X_1 \cup X_2 \cup V(C[x,y]))$ and $B = V(C[x,y])$. Note that the choice of P_0 implies that no long circuit of $G-(\{x,y\} \cup X_1 \cup X_2)$ intersects C only in B.

As in Case 1, there is a set X_3 of at most 3 vertices separating A and B in $G-(X_1 \cup X_2)$.

Arguing as in the proof of Claim B, the graph $G-(X_1 \cup X_2 \cup X_3)$ contains at most $\lceil \frac{1}{2}\ell \rceil + 1$ disjoint $(A, V(P_i))$-paths for each $i \in [2]$. This implies that if $G-(X_1 \cup X_2 \cup X_3)$ contains more than $\lceil \frac{1}{2}\ell \rceil + 1$ disjoint $(A, V(P_1) \cup V(P_2))$-paths, then one of these paths must end in $V(P_1') \setminus \{x,z\}$ and one of these paths must end in $V(P_2') \setminus \{x,z\}$. This immediately implies the existence of two disjoint parallel (A,B)-paths in $G-(X_1 \cup X_2)$, which is a contradiction. Hence there is a set X_4 of at most $\lceil \frac{1}{2}\ell \rceil + 1$ vertices separating A and $V(P_1) \cup V(P_2)$ in $G-(X_1 \cup X_2 \cup X_3)$.

Let $X = \{x,y\} \cup X_1 \cup X_2 \cup X_3 \cup X_4$.

If $G-X$ contains a long circuit, say D', then D' intersects A. Since D and D' intersect, there is an $(A, V(D))$-path P in $G-X$. In view of X_3, P cannot end in x and in view of X_4, P cannot end in $V(D) \setminus \{x\}$, which is a contradiction. Hence X intersects all long circuits. Clearly, as in Case 1, we have $|X| \leq \frac{3}{2}\ell + \frac{29}{2}$, which completes the proof in the second case. □

Proof of Theorem 1: Let C be shortest long circuit of G. Let L denote the length of C.

If L is at most $\frac{5}{3}\ell + \frac{29}{2}$, then let $X = V(C)$. If L is larger than $\frac{5}{3}\ell + \frac{29}{2}$ but less than $2\ell - 4$, then Lemma 2 implies the existence of a set X with the desired properties. If L is at least $2\ell - 3$, then Lemma 3 implies the existence of a set X with the desired properties. □

Our main interest was to improve the factor of ℓ in the bound in Theorem 1 and not the additive constant, which can easily be improved slightly.

The main open problem remains the conjectured inequality (3). Furthermore, it is unclear whether the quadratic dependence on k in (1) is best possible. For $\ell = 3$, that is, the classical case considered by Erdős and Pósa [3], it is known that $f_{\mathcal{F}_3}(k) = O(k \log k)$.

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