Cycle Lengths of Hamiltonian P_ℓ-free Graphs

Dirk Meierling, Dieter Rautenbach

Institut für Optimierung und Operations Research, Ulm University, Ulm, Germany

dirk.meierling@uni-ulm.de, dieter.rautenbach@uni-ulm.de

Abstract

For an integer ℓ at least three, we prove that every Hamiltonian P_ℓ-free graph G on $n > \ell$ vertices has cycles of at least $\frac{2}{\ell}n - 1$ different lengths. For small values of ℓ, we can improve the bound as follows. If $4 \leq \ell \leq 7$, then G has cycles of at least $\frac{1}{2}n - 1$ different lengths, and if ℓ is 4 or 5 and n is odd, then G has cycles of at least $n - \ell + 2$ different lengths.

Keywords: Cycle length; Cycle spectrum; Hamiltonian cycle

MSC2010: 05C38, 05C45

1 Introduction

In this paper we study the cycle spectrum of finite undirected graphs without loops or multiple edges and use standard notation and terminology as summarized in Subsection 1.1 below. The *cycle spectrum* $\mathcal{C}(G)$ *of a graph* G is the set of its cycle lengths and the cardinality of the cycle spectrum of G is denoted by $c(G)$.

Bondy's [2] so-called meta-conjecture states that essentially every non-trivial sufficient condition for the existence of a Hamiltonian cycle also implies pancyclicity, up to some well described exceptions. Bondy himself proved for example that every graph on n vertices that satisfies Ore's condition [9], that is, every pair of non-adjacent vertices has degree sum at least n, is pancyclic or the complete bipartite graph $K_{n/2,n/2}$. Using a result of Schmeichel and Hakimi [10], Bauer and Schmeichel [1] gave proofs that, with small families of exceptions, the conditions for Hamiltonian cycles due to Bondy [3], Chvátal [4], and Fan [5] also imply pancyclicity. Further results of this type may be found in [1, 10].

Conditions that are not strong enough to imply pancyclicity may still be strong enough to force a large cycle spectrum. At the 1999 conference "Paul Erdős and His Mathematics", Jacobson and Lehel asked how small the cycle spectrum of a graph G can be, when G only satisfies a relaxed version of a sufficient condition for the existence of a Hamiltonian cycle but is known to be Hamiltonian itself. Specifically, for an integer k with $3 \leq k \leq \lceil n/2 \rceil - 1$, they asked for the minimum cycle spectrum of a k-regular Hamiltonian graph on n vertices. Note that Bondy's result implies that all k-regular graphs of order n with $k \geq \lceil n/2 \rceil$, except for $K_{n/2,n/2}$, are pancyclic. Furthermore, 2-regular Hamiltonian graphs have exactly one cycle length. During the SIAM Meeting on Discrete Mathematics in 2002, Jacobson announced that he, Gould, and Pfender proved that every k-regular Hamiltonian graph G on n vertices has cycles of at least $c_k\sqrt{n}$ different lengths where $c_k > 0$ for $k \geq 3$. Milans et al. [6] strengthened the result by showing that every Hamiltonian graph with n vertices and m edges has cycles of at least $\sqrt{m-n} - \ln(m-n) - 1$ different lengths. In [7] Marczyk and Woźniak determine the minimum size of the cycle spectrum of Hamiltonian graphs in terms of their maximum degree. Recently, Müttel et al. [8] proved that cubic claw-free Hamiltonian graphs of order $n > 12$ have cycles of at least $\frac{1}{4}n + 3$ different lengths where a claw is a complete bipartite graph $K_{1,3}$.

If Q is a cycle or a path in a graph G, then a *chord of Q in G* is an edge of G that does not lie in Q but whose endvertices are in Q. The *length of a chord xy of Q* is equal to $\text{dist}_Q(x, y)$. Naturally, the existence, lengths, and distribution of chords of a Hamiltonian cycle is important for the existence of further cycles. The conditions mentioned above, such as degree conditions, regularity conditions, conditions on the size, or claw-freeness, all imply the existence of many chords and partly also concern their lengths and distribution.

In the present paper we consider P_ℓ-freeness as another natural condition that forces many well distributed chords in a Hamiltonian cycle. For an integer ℓ at least 3, a graph is *P_ℓ-free* if it contains no path of order ℓ as an induced subgraph.

Our results are as follows.

Theorem 1. *If G is a Hamiltonian P_ℓ-free graph on n vertices and $n > \ell \geq 8$, then $c(G) \geq \frac{2}{\ell}n - 1$.*

For small values of ℓ, we can improve this bound considerably.

Theorem 2. *If G is a Hamiltonian P_ℓ-free graph on n vertices with $n > \ell$ and $4 \leq \ell \leq 7$, then $c(G) \geq \frac{1}{2}n - 1$.*

For every even integer n, the complete bipartite graph $K_{n/2,n/2}$ is P_ℓ-free for every $\ell \geq 4$ and satisfies $c(K_{n/2,n/2}) = \frac{1}{2}n - 1$. Hence Theorem 2 is best possible. If G is a graph of odd order and ℓ is 4 or 5, then we can further improve the bound to $n - \ell + 2$, that is, these graphs are essentially pancyclic.

Theorem 3. *If G is a Hamiltonian P_4-free graph of odd order $n \geq 5$, then G is pancyclic.*

Theorem 4. *If G is a Hamiltonian P_5-free graph of odd order $n \geq 7$, then G has cycles of lengths $4, 5, \ldots, n$.*

Obviously, Theorem 3 is best possible. To see that the lower bound in Theorem 4 is sharp, consider the following family of graphs. For an even integer $n \geq 6$, subdivide an edge of the complete bipartite graph $K_{n/2,n/2}$ by a new vertex v. The resulting graph has odd order $n+1$, it is P_5-free, since every path of order 5 uses at least two vertices of each of the two partite sets of the original $K_{n/2,n/2}$, and it contains no 3-cycle, but cycles of lengths $4, 5, \ldots, n$.

We believe that the cycle spectrum of Hamiltonian P_ℓ-free graphs is much larger than guaranteed by Theorem 1. In view of Theorems 2 to 4 we venture to pose the following.

Conjecture 5. *If G is a Hamiltonian P_ℓ-free graph on n vertices, then $c(G) \geq \frac{1}{2}n - c_1$ and, if n is odd, $c(G) \geq n - c_2$ where c_1 and c_2 only depend on ℓ.*

After a summary of standard notation and terminology in the next subsection and some preliminaries in Section 2, we prove our results in Section 3.

1.1 Notation and Terminology

For a graph G, the *vertex set* is denoted by $V(G)$ and the *edge set* by $E(G)$. The *order* of G is denoted by $n(G)$ and its *size* by $m(G)$. For a positive integer k, a graph is *k-regular* if every vertex has exactly k neighbors, and a 3-regular graph is *cubic*. For a positive integer k, let $[k]$ denote the set $\{1, 2, \ldots, k\}$. A *path P in G of length ℓ between two vertices v_0 and v_ℓ* is a sequence of $\ell + 1$ distinct vertices $P = v_0 v_1 \cdots v_\ell$ such that $v_{i-1}v_i$ is an edge of G for every $i \in [\ell]$. Similarly, a *cycle C in G of length ℓ* with $\ell \geq 3$, or an *ℓ-cycle* for short, is a sequence $C = v_1 v_2 \cdots v_\ell v_1$ such that $v_1 v_2 \cdots v_\ell$ is a path in G and $v_\ell v_1$ is an edge of G. A *Hamiltonian cycle* is a cycle containing all vertices of a graph and a graph having such a cycle is called *Hamiltonian*. A graph on n vertices is *pancyclic* if its cycle spectrum is $\{3, 4, \ldots, n\}$. If P is a path and x and y are vertices of P, then $P[x, y]$ denotes the subpath of P between x and y. Similarly, if C is a cycle endowed with an orientation and x and y are vertices of C, then $C[x, y]$ denotes the segment of C from x to y following the orientation of C. For a graph G and two vertices v and w of G, the *distance* $\mathrm{dist}_G(v, w)$ *in G between v and w* is the minimum length of a path in G between v and w. If $W \subseteq V(G)$, then $G[W]$ denotes the subgraph of G induced by W. A graph that does not contain a graph H as an induced subgraph is called *H-free*.

2 Preliminaries

The observations and results in this section are frequently used to establish the base cases of our inductive proofs in Section 3.

Let $M = \{v_{r_i} v_{s_i} : i \in [k]\}$ be a matching consisting of chords of a cycle C. If the vertices incident with the edges in M appear on C in the cyclic order

- $v_{r_1}, v_{s_1}, v_{r_2}, v_{s_2}, \ldots, v_{r_k}, v_{s_k}$, then M is *independent*,
- $v_{r_1}, v_{r_2}, \ldots, v_{r_k}, v_{s_k}, v_{s_{k-1}}, \ldots, v_{s_1}$, then M is *parallel*, and
- $v_{r_1}, v_{r_2}, \ldots, v_{r_k}, v_{s_1}, v_{s_2}, \ldots, v_{s_k}$, then M is *crossing*.

Similarly, pairs of chords are *independent* if they are incident or form an independent set of chords. Otherwise, they are *crossing*. The following was observed by Müttel et al. [8].

Lemma 6 (Müttel et al. [8])**.** *Let C be a Hamiltonian cycle of a graph G. If C has an independent or parallel set of k chords, then $c(G) \geq k + 1$, and if C has a crossing set of k chords, then $c(G) \geq k$.*

The next two lemmas are obvious.

Lemma 7. *Let C be a Hamiltonian cycle of a graph G on n vertices. If C has a chord $v_r v_s$ of length t, then $C[v_r, v_s]v_r$ and $C[v_s, v_r]v_s$ are non-Hamiltonian cycles of lengths $t+1$ and $n-t+1$ where $t + 1 < n - t + 1$.*

Lemma 8. *Let C be a Hamiltonian cycle of a graph G on n vertices. If C has two crossing chords $v_{r_1} v_{s_1}, v_{r_2} v_{s_2}$, then G has the cycles*

$$C_1 = C[v_{r_1}, v_{r_2}]C[v_{s_2}, v_{s_1}]v_{r_1} \quad \textit{and} \quad C_2 = C[v_{r_1}, v_{s_2}]C[v_{r_2}, v_{s_1}]v_{r_1}$$

with $|V(C_1)| + |V(C_2)| = n + 4$.

Lemma 9. *Let C be a Hamiltonian cycle of a graph G of even order $n \geq 8$. If C has two chords of length $\frac{1}{2}n$, then $c(G) \geq 3$.*

Proof. By Lemma 7, G has a cycles of length $\frac{1}{2}n+1$ and, by Lemma 8, G has cycles of length k and $n-k+4$ where $4 \leq k \leq \frac{1}{2}n+2$. If $\{k, n-k+4\} \not\subseteq \{\frac{1}{2}n+1, n\}$, then $c(G) \geq 3$. Otherwise $\{k, n-k+4\} \subseteq \{\frac{1}{2}n+1, n\}$ and thus $n \leq 6$, a contradiction. □

Lemma 10. *Let C be a Hamiltonian cycle of a graph G of order $n \geq 6$. If C has two chords $v_{r_1}v_{s_1}, v_{r_2}v_{s_2}$ of length t with $t < \frac{1}{2}n$ and $3t-1 \neq n$, then $c(G) \geq 4$ or $t = 3$ and $G[\{v_{r_1}, v_{s_1}, v_{r_2}, v_{s_2}\}] = C_4$.*

Proof. If $v_{r_1}v_{s_1}, v_{r_2}v_{s_2}$ are independent, then G has cycles of length $t+1, n-2(t-1), n-(t-1), n$. Since $t < \frac{1}{2}n$, we have $t+1 < n-(t-1)$ and, since $3t-1 \neq n$, we have $t+1 \neq n-2(t-1)$. It follows that $c(G) \geq 4$.

So, assume that $v_{r_1}v_{s_1}, v_{r_2}v_{s_2}$ are crossing. By Lemma 7, G has cycles of lengths $t+1$ and $n-(t-1)$ with $t+1 < \frac{1}{2}n < n-(t-1)$, and, by Lemma 8, G has cycles of lengths k and $n-k+4$ with $k < \frac{1}{2}n+2 < n-k+4$.

If $\{k, n-k+4\} \not\subseteq \{t+1, n-t+1, n\}$, then $c(G) \geq 4$. So, assume that $\{k, n-k+4\} \subseteq \{t+1, n-t+1, n\}$. Now $n-k+4 = n$ and either $k = t+1$ or $k = n-t+1$. If $k = n-t+1$, then $n-3 = t < \frac{1}{2}n$, a contradiction to $n \geq 6$. Hence $k = t+1$ and thus $k = 4$ and $t = 3$. This implies that $G' = G[\{v_{r_1}, v_{s_1}, v_{r_2}, v_{s_2}\}]$ contains a 4-cycle. If G' has a further edge, then G has a 3-cycle, which implies $c(G) \geq 4$. This completes the proof. □

Lemma 11. *If G is a Hamiltonian P_ℓ-free graph on $n > \ell$ vertices, then every Hamiltonian cycle of G has at least two chords.*

Proof. Let $C = v_1v_2 \cdots v_nv_1$ be a Hamiltonian cycle of G. Since $n \geq \ell+1$, we may assume that C has the chord v_1v_r where $2 \leq r \leq n-1$. Then $C[v_2, v_n]$ is a path of order $n-1 \geq \ell$ and thus, forces a chord different from v_1v_r. □

3 Proofs of the theorems

In this section we prove Theorems 1 to 4.

3.1 Proof of Theorem 1

The proof is by induction on n. Let $C = v_1v_2 \cdots v_nv_1$ be a Hamiltonian cycle of G where we identify indices modulo n.

First let $\ell < n \leq 2\ell$. It suffices to show that $c(G) \geq 3$. By Lemma 11, C has at least two chords. If one of the chords is of length less than $\frac{1}{2}n$, then $c(G) \geq 3$ by Lemma 7, and if C has two chords of length $\frac{1}{2}n$, then $c(G) \geq 3$ by Lemma 9.

Now let $n > 2\ell$.

Since G is P_ℓ-free, C has a chord of length at most $\ell-1$, which implies that G contains a non-Hamiltonian cycle of length at least $n-\ell+1$.

If G has a non-Hamiltonian cycle C' of length $n' \geq n-\frac{1}{2}\ell$, then let G' be the graph induced by the vertices of C'. Note that G has at least one cycle length, namely n, that does not appear in G'. It follows, by induction, that

$$c(G) \geq 1 + c(G') \geq 1 + \frac{2}{\ell}n' - 1 \geq 1 + \frac{2}{\ell}\left(n - \frac{1}{2}\ell\right) - 1 = \frac{2}{\ell}n - 1.$$

So, assume that every non-Hamiltonian cycle of G is shorter than $n - \frac{1}{2}\ell$. This implies that every chord of C has length greater than $\frac{1}{2}\ell + 1$.

If G has a non-Hamiltonian cycle C' of length $n' \geq n - \ell$ and a non-Hamiltonian cycle of length $n'' > n'$, then let G' be the graph induced by the vertices of C'. Note that G has at least two cycle lengths, namely n and n'', that do not appear in G'. It follows, by induction, that

$$c(G) \geq 2 + c(G') \geq 2 + \frac{2}{\ell}n' - 1 \geq 2 + \frac{2}{\ell}(n - \ell) - 1 = \frac{2}{\ell}n - 1.$$

So, assume that G contains exactly one cycle length of the form $n - (t-1)$ in $[n - \ell, n - \frac{1}{2}\ell)$ and no cycle length in $[n - t + 2, n - 1]$. This implies that every chord of C is either of length t or of length at least $\ell + 2$. Note that $\frac{1}{2}\ell + 1 < t \leq \ell - 1$.

Let $v_{r_1}v_{s_1}$ be a chord of length t. Since $v_{r_1+2}v_{r_1+3}\cdots v_{r_1+\ell+1}$ is a path of order ℓ and $t > \frac{1}{2}\ell+1$, there exists a chord of length t that crosses $v_{r_1}v_{s_1}$. Among all such chords, let $v_{r_2}v_{s_2}$ be chosen such that $\mathrm{dist}_C(v_{r_1}, v_{r_2}) \geq 2$ is minimal. Note that the incident vertices appear in the cyclic order $v_{r_1}, v_{r_2}, v_{s_1}, v_{s_2}$ on C. Due to the choice of r_2 and the fact that G is P_ℓ-free, it follows that $|\{v_{r_1+2}, v_{r_1+2}, \ldots, v_{s_2-1}\}| \leq \ell - 1$. Hence, the cycle $v_{r_1}v_{s_1}v_{s_1-1}\cdots v_{r_2}v_{s_2}v_{s_2+1}\cdots v_{r_1}$ has length $k \geq n - \ell + 2$ and thus, $k = n - (t-1)$ by assumption. Since $v_{r_1}v_{s_1}$ and $v_{r_2}v_{s_2}$ are chords of length t, it follows that $\mathrm{dist}_C(v_{r_1}, v_{r_2}) = \frac{1}{2}(t+1)$. Note that if G contains the edge $v_{r_2+1}v_{s_2+1}$, then $v_{r_1}v_{s_1}v_{s_1-1}\cdots v_{r_2+1}v_{s_2+1}v_{s_2+2}\cdots v_{r_1}$ is a cycle of length $n - t - 1 \geq n - \ell$, a contradiction.

Iteratively repeating the same reasoning for $v_{r_i}v_{s_i}$ with $i \geq 2$ instead of $v_{r_1}v_{s_1}$, implies that v_iv_{i+1} is a chord of C if and only if $i - r_1$ is a multiple of $\frac{1}{2}(t+1)$. This also implies that n is a multiple of $\frac{1}{2}(t+1)$. Let $p = \frac{2n}{t+1} - 1$. Since $n > 2\ell$ and $t + 1 \leq \ell$, we have $p \geq 4$. Let $r_i = r_1 + \frac{1}{2}(t+1)(i-1)$ and $s_i = s_1 + \frac{1}{2}(t+1)(i-1)$. Note that $v_{(s_i)+1} = v_{r_{(i+2)}}$.

Since $\left\{v_{r_{2i}}v_{s_{2i}} : i \in \left[\left\lceil \frac{1}{2}p \right\rceil\right]\right\}$ is an independent set of chords, G contains cycles of lengths $n - (k-1)(t-1)$ for $k \in \left[\left\lceil \frac{1}{2}p \right\rceil + 1\right]$ (cycles of type 1). Furthermore, for every $k \in [p-1]$, the paths $C[v_{r_1}, v_{r_2}]$ and $C[v_{s_k}, v_{s_{k+1}}]$ together with the edges in $\{v_{s_i}v_{r_{(i+2)}} : i \in [k-1]\}$ and $\{v_{r_i}v_{s_i} : i \in [k+1]\}$ form a cycle of length $t + 2k + 1$ in G (cycles of type 2).

Since the longest cycle of type 2 has length $t+2(p-1)+1$ and $n-(q-1)(t-1) > t+2(p-1)+1$ for $q - 1 < \frac{t-3}{2(t-1)}(p-1)$ and $\frac{t-3}{2(t-1)}(p-1) < \lceil \frac{1}{2}p \rceil$, there are at least $\frac{t-3}{2(t-1)}(p-1)$ cycles of type 1 of distinct lengths whose lengths are all larger than the lengths of all cycles of type 2. Since the shortest cycle of type 2 has length $t + 3$ and G has a cycle of length $t + 1$ using exactly one chord of length t, we have

$$\begin{aligned}
c(G) &\geq \frac{t-3}{2(t-1)}(p-1) + (p-1) + 1 \\
&= \left(\frac{3}{2} - \frac{1}{t-1}\right)\left(\frac{2n}{t+1} - 2\right) + 1 \\
&> \left(\frac{3}{2} - \frac{2}{\ell}\right)\left(\frac{2n}{\ell} - 2\right) + 1 \\
&\geq \left(\frac{3}{2} - \frac{2}{8}\right)\left(\frac{2n}{\ell} - 2\right) + 1 \\
&= \frac{2n}{\ell} + \frac{n}{2\ell} - \frac{5}{2} + 1 \\
&\geq \frac{2n}{\ell} - \frac{1}{2},
\end{aligned}$$

which completes the proof. □

3.2 Proof of Theorem 2

The proof is by induction on n. Let $C = v_1v_2\cdots v_nv_1$ be a Hamiltonian cycle of G where we identify indices modulo n.

First let $\ell+1 \le n \le \ell+4$. By Lemma 11, C has at least two chords. Hence, if $n \in \{5,6\}$, then $c(G) \ge 2 \ge \frac{1}{2}n-1$. If $n \in \{7,8\}$, then $c(G) \ge 3 \ge \frac{1}{2}n-1$ by Lemmas 7 or 9. By Lemma 7, if C has two chords of lengths $t_1 < t_2 = \frac{1}{2}n$, then $c(G) \ge 4$, and if C has two chords of lengths $t_1 < t_2 < \frac{1}{2}n$, then $c(G) \ge 5$. Therefore, in all remaining cases ($\ell = 5$ and $n = 9$; $\ell = 6$ and $n \in \{9,10\}$; $\ell = 7$ and $n \in \{9,10,11\}$), we may assume that all chords of C have the same length.

First suppose that $n \in \{9,10\}$.

If all chords of C are of length 5, then $n = 10$ and we consider two cases depending on ℓ. If $\ell = 6$, then C has all possible chords of length 5 and thus, $c(G) \ge 4$ by Lemma 6. If $\ell = 7$, then C has three crossing chords of length 5. We may assume, without loss of generality, that $v_1v_6, v_2v_7, v_3v_8 \in E(G)$ or $v_1v_6, v_2v_7, v_4v_9 \in E(G)$. In both cases, G has cycles of lengths $4,6,8,10$.

If all chords are of length less than 5, then, by Lemma 10, we may assume v_1v_4 and v_2v_5 are chords of C. In particular, all chords of C have length 3. Since $C[v_3,v_9]$ is a path of order 7, it induces a chord of length 3. Another application of Lemma 10 yields $c(G) \ge 4$.

Next suppose that $n = 11$ and $\ell = 7$.

If a chord of C has length at most 3, then G has a non-Hamiltonian cycle C' of length $n' \ge n-2 = 9$ and the result follows from the above remarks concerning the cases $n \in \{9,10\}$.

If all chords of C have length 4, then we may assume that $v_1v_5 \in E(G)$. Note that G has cycles of lengths $5,8,11$. Since $C[v_2,v_8]$ and $C[v_3,v_9]$ are paths of order 7, it follows that $v_3v_7 \in E(G)$ or $v_4v_8 \in E(G)$ or $v_2v_6, v_5v_9 \in E(G)$. In the first two cases, G has cycles of lengths $6,9$, and in the latter case, G has cycles of lengths $4,7$. It follows that $c(G) \ge 5$.

If all chords of C have length 5, then C has three crossing chords of length 5. We may assume, without loss of generality, that $v_1v_6, v_2v_7, v_3v_8 \in E(G)$ or $v_1v_6, v_2v_7, v_4v_9 \in E(G)$ or $v_1v_6, v_2v_8, v_4v_{10} \in E(G)$ or $v_1v_6, v_2v_8, v_4v_9 \in E(G)$. In the first two cases, G has cycles of lengths $4,6,7,9,11$ and in the last two cases, G has cycles of lengths $5,6,7,10,11$.

Now let $n \ge \ell+5$. Since G is P_ℓ-free, C has a chord of length at most $\ell-1$.

If G has a non-Hamiltonian cycle C' of length $n' \ge n-2$, then let G' be the graph induced by the vertices of C'. Note that G has at least one cycle length, namely n, that does not appear in G'. It follows, by induction, that $c(G) \ge c(G')+1 \ge \frac{1}{2}n' \ge \frac{1}{2}n-1$. This completes the proof for $\ell = 4$. Furthermore, for $\ell \ge 5$, we may assume that every non-Hamiltonian cycle of G has at most $n-3$ vertices. In particular, every chord of C is of length at least 4.

If G has a non-Hamiltonian cycle C' of length $n' \ge n-4$ and a non-Hamiltonian cycle of length $n'' > n'$, then let G' be the graph induced by the vertices of C'. Note that G has at least two cycle lengths, namely n and n'', that do not appear in G'. It follows, by induction, that $c(G) \ge c(G')+2 \ge \frac{1}{2}n'+1 \ge \frac{1}{2}n-1$. So, assume that G contains at most one cycle length in $\{n-4, n-3\}$ and none in $\{n-2, n-1\}$.

Case 1: $\ell = 5$.

Since G is P_5-free, it follows that C has chords of length 4. Let v_rv_{r+4} be a such a chord. Since $C[v_{r+2}, v_{r+6}]$ is a path of order 5, C has the chord $v_{r+2}v_{r+6}$ and $v_rv_{r+4}v_{r+3}v_{r+2}v_{r+6}v_{r+7}\cdots v_r$ is a cycle of length $n-2$, a contradiction.

Case 2: $\ell = 6$.

Let $v_r v_{r+t}$ be a shortest chord of C. Since $C[v_{r+2}, v_{r+7}]$ is a path of order 6, it has a chord.

If $t = 5$, then C has no chord of length 4. Hence, C has the chord $v_{r+2}v_{r+7}$ and thus, contains the cycle $v_r v_{r+5} v_{r+4} v_{r+3} v_{r+2} v_{r+7} v_{r+8} \cdots v_r$ of length $n-2$, a contradiction.

If $t = 4$, then G has a cycle of length $n-3$. It follows that C has no chord of length 5, since G has only one cycle length in $\{n-4, n-3\}$. Hence, G contains at least one of $\{v_{r+2}v_{r+6}, v_{r+3}v_{r+7}\}$. In the first case, $v_r v_{r+4} v_{r+3} v_{r+2} v_{r+6} v_{r+7} \cdots v_r$ is a cycle of length $n-2$, a contradiction, and in the latter case, $v_r v_{r+4} v_{r+3} v_{r+7} v_{r+8} \cdots v_r$ is a cycle of length $n-4$, again a contradiction.

Case 3: $\ell = 7$.

Let $v_r v_{r+t}$ be a shortest chord of C. Since $C[v_{r+2}, v_{r+8}]$ is a path of order 7, it has a chord.

If $t = 4$, then G has a cycle of length $n-3$. It follows that G contains no cycles of lengths $n-4$, $n-2$ or $n-1$ and, in particular, no chords of lengths 2, 3 or 5. Hence, $C[v_{r+2}, v_{r+8}]$ has a chord of length 4 or 6. If G has the chord $v_{r+2}v_{r+8}$, then $v_r v_{r+4} v_{r+3} v_{r+2} v_{r+8} v_{r+9} \cdots v_r$ is a cycle of length $n-4$, a contradiction. Likewise, if G has the chord $v_{r+2}v_{r+6}$ or $v_{r+3}v_{r+7}$, then $v_r v_{r+4} v_{r+3} v_{r+2} v_{r+6} v_{r+7} \cdots v_r$ or $v_r v_{r+4} v_{r+3} v_{r+7} v_{r+8} \cdots v_r$ is a cycle of length $n-2$, again a contradiction. Hence, the only chord of $C[v_{r+2}, v_{r+8}]$ is $v_{r+4}v_{r+8}$. Since $C[v_{r+1}, v_{r+7}]$ is a path of order 7, it has a chord of length 4 or 6 incident to v_{r+1}. In the first case $v_{r+1} v_{r+5} v_{r+4} v_{r+8} v_{r+9} \cdots v_{r+1}$ is a cycle of length $n-4$, and in the latter case $v_{r+1} v_{r+7} v_{r+6} v_{r+5} v_{r+4} v_{r+8} v_{r+9} \cdots v_{r+1}$ is a cycle of length $n-2$. Both possibilities contradict our assumption.

If $t = 5$, then G has a cycle of length $n-4$. It follows that G contains no cycles of lengths $n-3$, $n-2$ or $n-1$ and, in particular, no chords of lengths 2, 3 or 4. Hence, $C[v_{r+2}, v_{r+8}]$ has a chord of length 5 or 6. If G has the chord $v_{r+2}v_{r+8}$, then $v_r v_{r+5} v_{r+4} v_{r+3} v_{r+2} v_{r+8} v_{r+9} \cdots v_r$ is a cycle of length $n-3$, a contradiction. Likewise, if G has the chord $v_{r+2}v_{r+7}$, then $v_r v_{r+5} v_{r+4} v_{r+3} v_{r+2} v_{r+7} v_{r+8} \cdots v_r$ is a cycle of length $n-2$, again a contradiction. Hence, the only chord of $C[v_{r+2}, v_{r+8}]$ is $v_{r+3}v_{r+8}$. Since $C[v_{r+1}, v_{r+7}]$ is a path of order 7, it has a chord of length 5 or 6 incident to v_{r+1}. In the first case $v_{r+1} v_{r+6} v_{r+5} v_{r+4} v_{r+3} v_{r+8} v_{r+9} \cdots v_{r+1}$ is a cycle of length $n-2$, and in the latter case $v_r v_{r+5} v_{r+4} v_{r+3} v_{r+2} v_{r+1} v_{r+7} v_{r+8} \cdots v_r$ is a cycle of length $n-1$. Both possibilities contradict our assumption.

If $t = 6$, then G contains no chords of lengths 2, 3, 4 or 5. Hence the only chord of $C[v_{r+2}, v_{r+8}]$ is $v_{r+2}v_{r+8}$. Now $v_r v_{r+6} v_{r+5} v_{r+4} v_{r+3} v_{r+2} v_{r+8} v_{r+9} \cdots v_r$ is a cycle of length $n-2$ in G. This final contradiction completes the proof. □

3.3 Proof of Theorem 3

The proof is by induction on n. Let $C = v_1 v_2 \cdots v_n v_1$ be a Hamiltonian cycle of G.

First let $n = 5$. Since C has a chord of length 3, G has cycles of lengths 3, 4, and 5.

Now let $n \geq 7$. We prove that G has a cycle of length $n-1$ and a cycle C' of length $n-2$. If G' is the subgraph induced by the vertices of C', then $c(G) \geq c(G') + 2 \geq n-2$ by induction.

Since G is P_4-free, C has a chord of length at most 3. If C has a chord of length 2 and a chord of length 3, then we are done. Similarly, if C has no chords of length 3, then C has two independent chords of length 2 and we are done. So, assume that C has no chords of length 2. This implies that G has all possible chords of length 3. Obviously, G has a cycle of length $n-2$. It remains to show that G has an $(n-1)$-cycle.

If $n \equiv 1 \mod 4$, then $v_1v_4v_5v_8v_9 \cdots v_{n-1}v_nv_3v_6v_7v_{10}v_{11} \cdots v_{n-3}v_{n-2}v_1$ is a cycle of length $n-1$. If $n \equiv 3 \mod 4$, then $v_1v_4v_5v_8v_9 \cdots v_{n-7}v_{n-6}v_{n-3}v_{n-4}v_{n-1}v_2v_3v_6v_7 \cdots v_{n-9}v_{n-8}v_{n-5}v_{n-2}v_1$ is a cycle of length $n-1$. □

3.4 Proof of Theorem 4

The proof is by induction on n. Let $C = v_1v_2 \cdots v_nv_1$ be a Hamiltonian cycle of G where we identify indices modulo n.

First let $n = 7$. By Lemma 11, C has at least two chords. If C has two chords of lengths 2 and 3, then, by Lemma 7, G is pancyclic. Hence, we may assume that all chords of C have the same length.

If all chords of C have length 2, then C has three independent chords, or a pair of independent and a pair of crossing chords. In both cases, G is pancyclic by Lemmas 6 and 8.

If all chords of C have length 3, then C has a pair of crossing chords. If $v_iv_{i+3}, v_{i+2}v_{i+4} \in E(G)$ for an index $i \in [7]$, then, by Lemmas 7 and 8, G has cycles of lengths $4, 5, 6, 7$. So, we may assume that $v_1v_4, v_2v_5 \in E(G)$ and $v_3v_6, v_3v_7, v_4v_7 \notin E(G)$. Now $C[v_3, v_7]$ is an induced path of order 5, a contradiction.

Now let $n \geq 9$. We prove that G has a cycle of length $n-1$ and a cycle of length $n-2$, which implies the result by induction. Since G is P_5-free, C has a chord of length at most 4.

Case 1: C has a chord of length 2.

If C has two independent chords of length 2 or a chord of length 2 and a chord of length 3, then G has cycles of lenghts $n-2$ and $n-1$.

If C has chords of length 2 and chords of length 4, but no chords of length 3, then G has a cycle of length $n-1$. We may assume that G has no two independent chords of length 2. Hence we may assume that $v_1v_3 \in E(G)$ and $v_2v_4 \notin E(G)$. Since $C[v_2, v_6]$ is a path of order 5, it follows that $v_2v_6 \in E(G)$. Therefore $v_1v_3v_2v_6v_7 \cdots v_1$ is a cycle of length $n-2$.

Case 2: C has no chords of length 2, but a chord of length 3.

In this case G has a cycle of length $n-2$ and it remains to show the existence of a cycle of length $n-1$.

First assume that C has a chord of length 4. If v_rv_{r+3} is a chord of length 3, then $C[v_{r+1}, v_{r+5}]$ is a path of order 5, which implies that G contains at least one of $\{v_{r+1}v_{r+5}, v_{r+1}v_{r+4}, v_{r+2}v_{r+5}\}$. Iteratively applying this observation implies that for some r, we have $v_rv_{r+3}, v_{r+1}v_{r+5} \in E(G)$ or $v_rv_{r+3}, v_{r+2}v_{r+5}, v_{r+2}v_{r+6} \in E(G)$. In the first case $v_rv_{r+3}v_{r+2}v_{r+1}v_{r+5} \cdots v_r$ is a cycle of length $n-1$ and in the second case $v_rv_{r+3}v_{r+4}v_{r+5}v_{r+2}v_{r+6} \cdots v_r$ is a cycle of length $n-1$.

Now we assume that C has no chord of length 4. Note that, for every chord xy of C, either $C[x, y]$ or $C[y, x]$ has even length. We assume that among all Hamiltonian cycles of G and all chords, C and xy are chosen such that the path of even length among $C[x, y]$ or $C[y, x]$, say $C[x, y]$, is shortest possible. We may assume that $C[x, y] = v_1v_2 \cdots v_{2q+1}$ for some $q \geq 3$. Since G is P_5-free, G contains at least one of $\{v_{n-1}v_2, v_nv_3\}$ and at least one of $\{v_{2q-1}v_{2q+2}, v_{2q}v_{2q+3}\}$.

Consider the paths $v_{2q-2}v_{2q-1}v_{2q}v_{2q+1}v_1$ and $v_4v_3v_2v_1v_{2q+1}$. Since C has no chords of lengths 2 or 4, the choice of C and xy implies that G contains at least one of $\{v_{2q-2}v_{2q+1}, v_1v_{2q}, v_1v_{2q-2}\}$ and at least one of $\{v_1v_4, v_2v_{2q+1}, v_4v_{2q+1}\}$. If $v_1v_{2q} \in E(G)$, then $v_{2q}v_1v_{2q+1}$, $ab \in \{v_{n-2}v_2, v_nv_3\}$, $C[b, v_{2q}]$, and $C[v_{2q+1}, a]$ form an $(n-1)$-cycle in G. By symmetry, we may assume that $v_1v_{2q}, v_2v_{2q+1} \notin E(G)$.

If $v_1v_4, v_nv_3 \in E(G)$, then $C' = v_nv_3v_2v_1C[v_4, v_n]$ is a Hamiltonian cycle of G with the chord v_1v_{2q+1}. Now $C'[v_1, v_{2q+1}]$ is of even length less than $2q$, a contradiction to the choice of C and xy.

If $v_1v_4, v_{n-1}v_2 \in E(G)$, but $v_nv_3 \notin E(G)$, then consider the path $v_3v_2v_1v_{2q+1}v_{2q}$ of order 5. Since G is P_5-free, it has a chord. The edges v_2v_{2q+1} and v_1v_{2q} are not in G. Furthermore, by the choice of C and xy, the edges v_2v_{2q} and v_3v_{2q+1} are not in G. Finally, since C has no chord of length 2, the edge v_1v_3 is not in G. Hence, $v_3v_{2q} \in E(G)$ and thus $v_{n-1}v_2v_3v_{2q}v_{2q-1} \cdots v_4v_1C[v_{2q+1}, v_{n-1}]$ is an $(n-1)$-cycle in G. By symmetry, we may assume that $v_1v_4, v_{2q-2}v_{2q+1} \notin E(G)$.

This implies $v_1v_{2q-2}, v_4v_{2q+1} \in E(G)$. In view of the paths $v_nv_1 \cdots v_4$ and $v_1 \cdots v_5$, it follows that $v_nv_3, v_2v_5 \in E(G)$. Now $v_nv_3v_4v_{2q+1}v_1v_2v_5$, $C[v_5, a]$, $ab \in \{v_{2q-1}v_{2q+2}, v_{2q}v_{2q+3}\}$, and $C[b, v_n]$ form an $(n-1)$-cycle in G, which completes the proof in this case.

Case 3: C has no chords of length 2 and 3.

In this case C has all possible chords of length 4. It follows that $v_1v_5v_6v_7v_3v_4v_8v_9 \cdots v_1$ is an $(n-1)$-cycle and $v_1v_5v_4v_3v_7v_8 \cdots v_1$ is an $(n-2)$-cycle in G.

This completes the proof of the theorem. □

Acknowledgment

We were supported by the DFG project "Cycle Spectra of Graphs" RA873/5-1.

References

[1] D. Bauer and E. F. Schmeichel, Hamiltonian degree conditions which imply a graph is pancyclic. *J. Combin. Theory (B)* **48** (1990), 111–116.

[2] J. A. Bondy, Pancyclic Graphs. *J. Combin. Theory (B)* **11** (1971), 80–84.

[3] J. A. Bondy, Longest paths and cycles in graphs of high degree. Res. Rep. No. CORR 80-16, Univ. Waterloo, Waterloo, ON, 1980.

[4] V. Chvátal, On Hamiltons ideals. *J. Combin. Theory (B)* **12** (1972), 163–168.

[5] G. Fan, New sufficient conditions for cycles in graphs. *J. Combin. Theory (B)* **37** (1984), 221–227.

[6] K. G. Milans, F. Pfender, D. Rautenbach, F. Regen, D. B. West, Cycle spectra of Hamiltonian graphs. *J. Combin. Theory (B)* **102** (2012), 869–874.

[7] A. Marczyk, M. Woźniak, Cycles in hamiltonian graphs of prescribed maximum degree, *Discrete Math.* **266** (2003), 321–326.

[8] J. Müttel, D. Rautenbach, F. Regen, T. Sasse, On the Cycle Spectrum of Cubic Hamiltonian Graphs. *Graphs Comb.* doi:10.1007/s00373-012-1156-0.

[9] O. Ore, Note on Hamilton circuits. *Amer. Math. Monthly* **67** (1960), 55.

[10] E. F. Schmeichel and S. L. Hakimi, A cycle structure theorem for Hamiltonian graphs. *J. Combin. Theory (B)* **45** (1988), 99–107.