Nordhaus-Gaddum bounds on the k-rainbow domatic number of a graph

[1] D. Meierling, [2,3]S.M. Sheikholeslami, and [1]L. Volkmann

[1]Lehrstuhl II für Mathematik
RWTH Aachen University
52056 Aachen, Germany

meierling@math2.rwth-aachen.de
volkm@math2.rwth-aachen.de

[2]Department of Mathematics
Azarbaijan University of Tarbiat Moallem
Tabriz, I.R. Iran

[3]School of Mathematics
Institute for Research in Fundamental Sciences (IPM)
P.O. Box: 19395-5746, Tehran, I.R. Iran

s.m.sheikholeslami@azaruniv.edu

Abstract

For a positive integer k, a *k-rainbow dominating function* of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2, \dots, k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2, \dots, k\}$ is fulfilled, where $N(v)$ is the neighborhood of v. The 1-rainbow domination is the same as the ordinary domination. A set $\{f_1, f_2, \dots, f_d\}$ of k-rainbow dominating functions on G with the property that $\sum_{i=1}^{d} |f_i(v)| \le k$ for each $v \in V(G)$, is called a *k-rainbow dominating family* (of functions) on G. The maximum number of functions in a k-rainbow dominating family on G is the *k-rainbow domatic number* of G, denoted by $d_{rk}(G)$. Note that $d_{r1}(G)$ is the classical domatic number $d(G)$. If G is a graph of order n and $\overline{G}$ is the complement of G, then we prove in this note for $k \ge 2$ the Nordhaus-Gaddum inequality

$$d_{rk}(G) + d_{rk}(\overline{G}) \le n + 2k - 2.$$

This improves the Nordhaus-Gaddum bound given by Sheikholeslami and Volkmann recently.

Keywords: k-rainbow dominating function, k-rainbow domination number, k-rainbow domatic number, Nordhaus-Gaddum bound.

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1 Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \colon uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *open neighborhood* of a set $S \subseteq V$ is the set

$N(S) = \bigcup_{v\in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. The complement of a graph G is denoted by $\overline{G}$. We write K_n for the *complete graph* of order n and C_n for a *cycle* of length n. Consult [7, 10] for notation and terminology which are not defined here.

A subset S of vertices of G is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A domatic partition is a partition of V into dominating sets, and the domatic number $d(G)$ is the largest number of sets in a domatic partition. The domatic number was introduced by Cockayne and Hedetniemi [5]. In their paper, they showed that $\gamma(G) \cdot d(G) \le n$.

For a positive integer k, a *k-rainbow dominating function* (kRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u\in N(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v\in V} |f(v)|$. The *$k$-rainbow domination number* of a graph G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a kRDF of G. A *$\gamma_{rk}(G)$-function* is a k-rainbow dominating function of G with weight $\gamma_{rk}(G)$. Note that $\gamma_{r1}(G)$ is the classical domination number $\gamma(G)$. The k-rainbow domination number was introduced by Brešar, Henning, and Rall [1] and has been studied by several authors (see for example [2, 3, 4, 11]). Rainbow domination of a graph G coincides with ordinary domination of the Cartesian product of G with the complete graph, in particular, $\gamma_{rk}(G) = \gamma(G \square K_k)$ for any graph G [1]. This implies (cf. [3]) that

$$\gamma_{r1}(G) \le \gamma_{r2}(G) \le \cdots \le \gamma_{rk}(G) \le n$$

for any graph G of order n. Furthermore, it was proved in [6] that

$$\min\{|V(G)|, \gamma(G) + k - 2\} \le \gamma_{rk}(G) \le k\gamma(G)$$

for any $k \ge 2$ and any graph G. Combining the inequality $\gamma(G) \ge \lceil \frac{n}{\Delta+1} \rceil$ given in [9] and the identity $\gamma_{rk}(G) = \gamma(G \square K_k)$ given in [1], we obtain the following lower bound for the k-rainbow domination number of a graph G. If G is a graph of order n and maximum degree Δ, then

$$\gamma_{rk}(G) \ge \left\lceil \frac{kn}{\Delta + k} \right\rceil.$$

(Another direct proof of this inequality is given in the first part of the proof of Theorem 7: In an arbitrary graph G the inequalities (2) and (3) are valid if we replace δ by Δ.)

A set $\{f_1, f_2, \ldots, f_d\}$ of k-rainbow dominating functions of a graph G with the property that $\sum_{i=1}^{d} |f_i(v)| \le k$ for each $v \in V(G)$, is called a *k-rainbow dominating family* (of functions) on G. The maximum number of functions in a k-rainbow dominating family (kRD family) on G is the *k-rainbow domatic number* of G, denoted by $d_{rk}(G)$. The k-rainbow domatic number is well-defined and

$$d_{rk}(G) \ge k \qquad (1)$$

for all graphs G, since the set consisting of the functions $f_i \colon V(G) \to \mathcal{P}(\{1, 2, \ldots, k\})$ defined by $f_i(v) = \{i\}$ for each $v \in V(G)$ and each $i \in \{1, 2, \ldots, k\}$, forms a kRD family on G. Note that $d_{r1}(G)$ is the classical domatic number $d(G)$.

The k-rainbow domatic number was introduced and investigated by Sheikholeslami and Volkmann [8]. The following results on the k-rainbow domatic number are important for our investigations.

Theorem 1 (Sheikholeslami & Volkmann [8]). *For every graph G with minimum degree δ,*

$$d_{rk}(G) \le \delta + k.$$

Theorem 2 (Sheikholeslami & Volkmann [8]). *For every graph G of order n,*

$$d_{rk}(G) \le n.$$

The special case $k = 1$ in Theorem 1 was done by Cockayne and Hedetniemi [5]. As an application of Theorem 1, Sheikholeslami and Volkmann proved the following Nordhaus-Gaddum type result.

Theorem 3 (Sheikholeslami & Volkmann [8])**.** *For every graph G of order n,*

$$d_{rk}(G) + d_{rk}(\overline{G}) \leq n + 2k - 1.$$

If $d_{rk}(G) + d_{rk}(\overline{G}) = n + 2k - 1$, then G is regular.

Corollary 4 (Cockayne & Hedetniemi [5] 1977)**.** *If G is a graph of order n, then $d(G)+d(\overline{G}) \leq n+1$.*

Theorem 5 (Sheikholeslami & Volkmann [8])**.** *If k is a positive integer, and G is isomorphic to the complete graph K_n of order $n \geq k$, then $d_{rk}(G) = n$.*

In their paper [8], the authors posed the following conjecture.

Conjecture 1. *For every integer $k \geq 2$ and every graph G of order n,*

$$d_{rk}(G) + d_{rk}(\overline{G}) \leq n + 2k - 2.$$

The purpose of this note is to prove the aforementioned conjecture.

2 Nordhaus-Gaddum bounds

Using (1), our first Nordhaus-Gaddum inequality is immediate.

Theorem 6. *If $k \geq 1$ is an integer, and G is a graph of order n, then*

$$2k \leq d_{rk}(G) + d_{rk}(\overline{G}).$$

The next result gives an upper bound for the k-rainbow domatic number of some special regular graphs.

Theorem 7. *Let G be a δ-regular graph of order n. If G has a $\gamma_{rk}(G)$-function f such that $V_2 \cup V_3 \cup \cdots \cup V_k \neq \emptyset$ or $V_2 = V_3 = \cdots = V_k = \emptyset$ and $k|V_0| < \delta|V_1|$, where $V_i = \{v \in V(G) \colon |f(v)| = i\}$, then*

$$d_{rk}(G) \leq \delta + k - 1.$$

Proof. Let f be a $\gamma_{rk}(G)$-function and let $V_i = \{v \colon |f(v)| = i\}$ for $i = 0, 1, \ldots, k$. Then $\gamma_{rk}(G) = |V_1| + 2|V_2| + \cdots + k|V_k|$ and $n = |V_0| + |V_1| + \cdots + |V_k|$. Let $E_0 = (V_0, V \setminus V_0)$ be the edges from V_0 to $V \setminus V_0$. Since f is a $\gamma_{rk}(G)$-function, we obtain

$$k|V_0| \leq \sum_{xy \in E_0,\ x \in V \setminus V_0} |f(x)| \leq \delta(|V_1| + 2|V_2| + \cdots + k|V_k|) = \delta\gamma_{rk}(G). \tag{2}$$

Now it follows from (2) that

$$\begin{aligned}(\delta + k)\gamma_{rk}(G) &= \delta\gamma_{rk}(G) + k\gamma_{rk}(G) \\ &\geq k|V_0| + k(|V_1| + 2|V_2| + \cdots + k|V_k|) \\ &= k(|V_0| + |V_1| + \cdots + |V_k|) + k(|V_2| + 2|V_3| + \cdots + (k-1)|V_k|) \\ &= kn + k(|V_2| + 2|V_3| + \cdots + (k-1)|V_k|) \\ &\geq kn.\end{aligned} \tag{3}$$

Let $\{f_1, f_2, \ldots, f_d\}$ be a kRD family of G such that $d = d_{rk}(G)$. It follows that

$$\sum_{i=1}^{d} \omega(f_i) = \sum_{i=1}^{d} \sum_{v \in V} |f_i(v)| = \sum_{v \in V} \sum_{i=1}^{d} |f_i(v)| \leq \sum_{v \in V} k = kn. \tag{4}$$

Suppose to the contrary that $d \geq \delta + k$. If $V_2 \cup V_3 \cup \cdots \cup V_k \neq \emptyset$, then (3) shows that $\gamma_{rk}(G) \geq (kn+k)/(\delta+k)$. It follows that

$$\sum_{i=1}^{d} \omega(f_i) \geq \sum_{i=1}^{d} \gamma_{rk}(G) \geq d\left\lceil \frac{kn+k}{\delta+k} \right\rceil \geq (\delta+k)\left(\frac{kn+k}{\delta+k}\right) = kn+k > kn,$$

a contradiction to (4). If $V_2 = V_3 = \cdots = V_k = \emptyset$ and $k|V_0| < \delta|V_1|$, then $\gamma_{rk}(G) = |V_1|$ and $n = |V_0| + |V_1|$ and thus

$$(\delta+k)\gamma_{rk}(G) = k|V_1| + \delta|V_1| > k|V_1| + k|V_0| = kn.$$

This implies that $\gamma_{rk}(G) > kn/(\delta+k)$, and we obtain the following contradiction to (4)

$$\sum_{i=1}^{d} \omega(f_i) \geq \sum_{i=1}^{d} \gamma_{rk}(G) > (\delta+k)\left(\frac{kn}{\delta+k}\right) = kn.$$

Therefore $d \leq \delta + k - 1$, and the proof is complete. □

Now we improve the upper bound given in Theorem 3 for $k \geq 2$.

Theorem 8. *If $k \geq 2$ is an integer, and G is a graph of order n, then*

$$d_{rk}(G) + d_{rk}(\overline{G}) \leq n + 2k - 2.$$

Proof. If G is not regular, then Theorem 3 implies the desired result. Now let G be δ-regular.

Assume that G has a $\gamma_{rk}(G)$-function f such that $V_2 \cup V_3 \cup \cdots \cup V_k \neq \emptyset$ or $V_2 = V_3 = \cdots = V_k = \emptyset$ and $k|V_0| < \delta|V_1|$, where $V_i = \{v \in V(G) \colon |f(v)| = i\}$. Then we deduce from Theorem 7 that $d_{rk}(G) \leq \delta + k - 1$. Using Theorem 1, we obtain the desired result as follows

$$\begin{aligned} d_{rk}(G) + d_{rk}(\overline{G}) &\leq (\delta(G) + k - 1) + (\delta(\overline{G}) + k) \\ &= (\delta(G) + k - 1) + (n - \delta(G) - 1 + k) \\ &= n + 2k - 2. \end{aligned}$$

It remains the case that every $\gamma_{rk}(G)$-function f of G fulfills $V_2 = V_3 = \cdots = V_k = \emptyset$ and $k|V_0| = \delta|V_1|$. Note that $n = |V_0| + |V_1|$. Furthermore, $|V_0| \geq 1$ and thus $|V_1| \geq k$. Since $\delta(G) + \delta(\overline{G}) = n - 1$, it follows that $\delta(G) \geq (n-1)/2$ or $\delta(\overline{G}) \geq (n-1)/2$. We assume, without loss of generality, that $\delta(G) \geq (n-1)/2$.

If $|V_1| \geq 2k$, then $k|V_0| = \delta|V_1| \geq 2k\delta$ and thus $|V_0| \geq 2\delta$. This leads to the contradiction

$$n = |V_0| + |V_1| \geq 2\delta + 2k \geq n - 1 + 2k.$$

In the case $k + 1 \leq |V_1| \leq 2k - 1$, we define $V_1^i = \{v \colon f(v) = \{i\}\}$ for $i \in \{1, 2, \ldots, k\}$. Because of $|V_1| \leq 2k - 1$, we observe that $|V_1^i| = 1$ for at least one index $i \in \{1, 2, \ldots, k\}$. We assume, without loss of generality, that $|V_1^1| = 1$. Since each vertex of V_0 is adjacent to the vertex of V_1^1, we deduce that $|V_0| \leq \delta$. This implies that

$$k|V_0| \leq k\delta < \delta|V_1|,$$

a contradiction to the assumption $k|V_0| = \delta|V_1|$.

If $|V_1| = k$, then $|V_0| = \delta$ and so $n = \delta + k$. Hence $\delta(\overline{G}) = n - \delta - 1 = k - 1$. Since the k vertices of V_1 induce a complete component of order k in $\overline{G}$, we deduce from Theorem 5 that $d_{rk}(\overline{G}) \leq k$. Now Theorem 1 implies that

$$d_{rk}(G) + d_{rk}(\overline{G}) \leq (\delta(G) + k) + k = n + k \leq n + 2k - 2.$$

Since we have discussed all possible cases, the proof is complete. □

Corollary 9. *If $k \geq 2$ is an integer, and G is a graph of order n, then*

$$d_{rk}(G) \cdot d_{rk}(\overline{G}) \leq \frac{(n+2k-2)^2}{4}.$$

Proof. It follows from Theorem 8 that

$$\begin{aligned} (n+2k-2)^2 &\geq (d_{rk}(G) + d_{rk}(\overline{G}))^2 \\ &= (d_{rk}(G) - d_{rk}(\overline{G}))^2 + 4d_{rk}(G) \cdot d_{rk}(\overline{G}) \\ &\geq 4d_{rk}(G) \cdot d_{rk}(\overline{G}) \end{aligned}$$

and this leads to the desired bound. □

For the special case $k = 2$, Theorem 8 was proved by Sheikholeslami and Volkmann [8]. The complete graph K_n demonstrates that Theorem 8 does not hold for $k = 1$. Consider the complete p-partite graph $G = K_{k,k,\ldots,k}$ of order $n = pk$. Let $V(G) = \{v_i^j : i = 1, 2, \ldots, k; j = 1, 2, \ldots, p\}$ and $E(G) = \{v_i^j v_s^t : 1 \leq j \neq t \leq p; i, s = 1, 2, \ldots, k\}$. For $\ell = 1, 2, \ldots, n$ we define $f_\ell(v_i^j)$ as follows: Write $\ell + i = qk + r$, where $0 \leq r \leq k-1$, and set $f_\ell(v_i^j) = r + 1$ if $\lceil \ell/k \rceil = j$, and $f_\ell(v_i^j) = \emptyset$ otherwise. Then $\{f_1, f_2, \ldots, f_n\}$ is a kRD family on G. In view of Theorem 2, it follows that $d_{rk}(K_{k,k,\ldots,k}) = n$. Since $\overline{K_{k,k,\ldots,k}}$ consists of p complete graphs each of order k, it holds $d_{rk}(\overline{K_{k,k,\ldots,k}}) = k$ and thus,

$$d_{rk}(K_{k,k,\ldots,k}) + d_{rk}(\overline{K_{k,k,\ldots,k}}) = n + k.$$

Hence the complete p-partite graph $K_{2,2,\ldots,2}$ shows that the bound in Theorem 8 is best possible for $k = 2$. Furthermore, we conjecture the following.

Conjecture 2. *If $k \geq 2$ is an integer, and G is a graph of order n, then*

$$d_{rk}(G) + d_{rk}(\overline{G}) \leq n + k.$$

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