The signed domatic number of some regular graphs

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Abstract

Let G be a finite and simple graph with vertex set $V(G)$, and let $f : V(G) \to \{-1, 1\}$ be a two-valued function. If $\sum_{x \in N[v]} f(x) \geq 1$ for each $v \in V(G)$, where $N[v]$ is the closed neighborhood of v, then f is a signed dominating function on G. A set $\{f_1, f_2, \ldots, f_d\}$ of signed dominating functions on G with the property that $\sum_{i=1}^{d} f_i(x) \leq 1$ for each $x \in V(G)$, is called a signed dominating family (of functions) on G. The maximum number of functions in a signed dominating family on G is the signed domatic number on G. In this paper we investigate the signed domatic number of some circulant graphs and of the torus $C_p \times C_q$.

Keywords: Signed domatic number, Signed dominating function, Signed domination number, Circulant graphs, Torus graph.

AMS Subject Classification: 05C69

1 Terminology and introduction

We consider finite, undirected and simple graphs G with vertex set $V(G)$. If v is a vertex of the graph G, then $N(v) = N_G(v)$ is the *open neighborhood* of v, i.e., the set of all vertices adjacent with v. The *closed neighborhood* $N[v] = N_G[v]$ of a vertex v consists of the vertex set $N(v) \cup \{v\}$. The number $d_G(v) = d(v) = |N(v)|$ is the *degree* of the vertex $v \in V(G)$, and $\delta(G)$ is the *minimum degree* of G. The *cycle* of order n is denoted by C_n. If $A \subseteq V(G)$ and f is a mapping from $V(G)$ into some set of numbers, then $f(A) = \sum_{x \in A} f(x)$.

The *signed dominating function* is defined in [2] as a two-valued function $f : V(G) \to \{-1, 1\}$ such that $\sum_{x \in N[v]} f(x) \geq 1$ for each $v \in V(G)$. The sum $f(V(G))$ is called the weight $w(f)$ of f. The minimum of weights $w(f)$, taken over all signed dominating functions f on G, is called the *signed domination number* of G, denoted by $\gamma_S(G)$. Signed domination has been studied in [2], [3], [4], [7], [8] and [11]. Further information on this parameter can be found in the monographs [5] and [6] by Haynes, Hedetniemi and Slater.

A set $\{f_1, f_2, \ldots, f_d\}$ of signed dominating functions on G with the property that $\sum_{i=1}^{d} f_i(x) \le 1$ for each vertex $x \in V(G)$, is called a *signed dominating family* on G. The maximum number of functions in a signed dominating family on G is the *signed domatic number* of G, denoted by $d_S(G)$. The signed domatic number was introduced by Volkmann and Zelinka [10]. Volkmann and Zelinka [10] and Volkmann [9] have determined the signed domatic number of complete graphs and complete bipartite graphs, respectively. In addition, Volkmann and Zelinka [10] presented the following two basic results, which are useful for our investigations.

Theorem 1.1 (Volkmann, Zelinka [10] 2005) If G is a graph, then

$$1 \le d_S(G) \le \delta(G) + 1.$$

Theorem 1.2 (Volkmann, Zelinka [10] 2005) The signed domatic number is an odd integer.

Next we derive a structural result on $2r$-regular graphs with maximal possible signed domatic number.

Theorem 1.3 Let G be a $2r$-regular graph, and let u be an arbitrary vertex of G. If $d = d_S(G) = 2r + 1$ and $\{f_1, f_2, \ldots, f_d\}$ is a signed domatic family of G, then $\sum_{i=1}^{d} f_i(u) = 1$ and $\sum_{x \in N[u]} f_i(x) = 1$ for each $u \in V(G)$ and each $i \in \{1, 2, \ldots, 2r+1\}$.

Proof. Let u be an arbitrary vertex of G. Because of $\sum_{i=1}^{d} f_i(u) \le 1$, this sum contains at least r summands which have the value -1. Using the fact that $\sum_{x \in N[u]} f_i(x) \ge 1$ for each $i \in \{1, 2, \ldots, 2r+1\}$, we observe that each of these sums contains at least $r + 1$ summands which have the value 1. Consequently, the sum

$$\sum_{x \in N[u]} \sum_{i=1}^{d} f_i(x) = \sum_{i=1}^{d} \sum_{x \in N[u]} f_i(x) \tag{1}$$

contains at least dr summands of value -1 and at least $d(r+1)$ summands of value 1. As the sum (1) consists of exactly $d(2r+1)$ summands, we conclude that $\sum_{i=1}^{d} f_i(u)$ contains exactly r summands of value -1 and $\sum_{x \in N[u]} f_i(x)$ contains exactly $r+1$ summands of value 1 for each $i \in \{1, 2, \ldots, 2r+1\}$. This leads to the desired result, and the proof is complete. □

2 Circulant graphs

Following an article of Boesch and Tindell [1], for an integer $n \geq 3$ and a subset S of $\{1, 2, \ldots, \lfloor (n+2)/2 \rfloor\}$, the *circulant graph* $C_n(S)$ is a graph on n vertices $u_1, u_2, \ldots, u_n$ such that each vertex u_i is adjacent to the the vertices $u_{i \pm s}$ for $s \in S$, where the subscripts are taken modulo n. Certainly, $C_n(\{1\})$ is isomorphic to the cycle C_n and $C_n(\{1, 2\})$ is isomorphic to the square C_n^2 of C_n. It is easy to observe that circulant graphs are vertex-symmetric.

Theorem 2.1 If G is the circulant graph $C_n(\{1, 2, \ldots, r\})$, then $d_S(G) = 2r + 1$ if and only if $n \equiv 0 \pmod{2r+1}$.

Proof. Let G be the circulant graph $C_n(\{1, 2, \ldots, r\})$ with vertex set $\{u_1, u_2, \ldots, u_n\}$. Since G is $2r$-regular, Theorem 1.1 implies that $d_S(G) \leq 2r + 1$.

Suppose that $n \equiv 0 \pmod{2r+1}$. In this case, we define a signed dominating family $\{f_1, f_2, \ldots, f_{2r+1}\}$ as follows:

$$f_i(u_i) = f_i(u_{i+1}) = \ldots = f_i(u_{i+r-1}) = -1$$

and

$$f_i(u_{i+r}) = f_i(u_{i+r+1}) = \ldots = f_i(u_{i+2r}) = 1$$

for $i = 1, 2, \ldots, 2r+1$, where the indices of the vertices are taken modulo $2r+1$. For the remaining vertices u_{2r+1+k} with $k \geq 1$, we define the function f_i by $f_i(u_{2r+1+k}) = f_i(u_k)$ for $i \in \{1, 2, \ldots, 2r+1\}$.

Suppose that $n = k(2r+1) + s$ with $s \in \{1, 2, \ldots, 2r\}$. Assume that $d_S(G) = d = 2r+1$ and let $\{f_1, f_2, \ldots, f_d\}$ be a corresponding signed dominating family. Let

$$A_0 = \{i : i \equiv j(2r+1) \pmod{n}, j \in \mathbb{N}_0\}$$

be the set of indices i we derive from 0 by adding a multiple of $2r+1$ modulo n. Let t be the greatest common divisor of s and $2r+1$, i.e., let $s = pt$ and $2r+1 = qt$ with p and q relatively prime. Let $a \geq 1$ and $b \geq 1$ be the smallest integers such that $a(2r+1) = bn = b[k(2r+1) + s]$. Then

$$a(2r+1) = b[k(2r+1) + s] \Leftrightarrow aqt = bt(kq + p) \Leftrightarrow aq = b(kq + p).$$

Since p and q are relatively prime, q divides b. Analogously, $kq + p$ divides a. Hence, $a = kq + p$ and $b = q$. It follows that $|A_0| = kq + p$. Analogously, for every $\ell \in \{1, 2 \ldots, 2r\}$, the set $A_\ell = \{j : j \equiv \ell + i \pmod{n}, i \in A_0\}$ contains exactly $kq + p$ elements. Therefore, the set $\{1, 2, \ldots, n\}$ can be partitioned in t sets $A_{j_1}, A_{j_2}, \ldots, A_{j_t}$ of size $kq + p$. Note that, by Theorem 1.3, $\sum_{i=1}^{d} f_i(u_j)$ contains exactly r summands of value -1 for each $j \in \{1, 2, \ldots, n\}$ and $\sum_{x \in N[u_j]} f_i(x)$ contains exactly $r + 1$ summands of value 1 for each $j \in \{1, 2, \ldots, n\}$ and $i \in \{1, 2, \ldots, 2r+1\}$. It follows that $f_i(v) = f_i(w)$ for every pair of vertices $\{v, w\} \subseteq A_\ell$, every $i \in \{1, 2, \ldots, d\}$ and every $\ell \in \{0, 1, \ldots, 2r\}$. Furthermore, note that each set A_ℓ contains exactly q elements of $\{1,2,\dots,2r+1\}$, by definition. This implies that

$$1 = \sum_{x\in N[u_{r+1}]} f_i(x) = t_1 q - t_2 q = q(t_1 - t_2)$$

with $t_1 + t_2 = t$ and $t_1 > t_2$ for an arbitrary $i \in \{1,2,\dots,d\}$. Hence, $q = 1$, a contradiction to $pt = s < 2r+1 = qt$. So $d < 2r+1$, and the proof is complete. □

The least common multiple $\mathrm{lcm}(a,b)$ of two integers a and b is the smallest number greater than zero which is divisible by a and b.

Theorem 2.2 Let G be the circulant graph $C_n(\{d, 2d, \dots, rd\})$. Then $d_S(G) = 2r+1$ if and only if $\frac{\mathrm{lcm}(n,d)}{d} \equiv 0 \pmod{2r+1}$.

Proof. First we dicuss the case $r = 1$. For an arbitrary vertex u_1 we investigate the structure of the component F_1 containing u_1. Clearly, $u_1 \in V(F_1)$, and since u_1 is adjacent to u_{d+1}, it follows that $u_{d+1} \in V(F_1)$. The vertex u_{d+1} is adjacent to u_{2d+1}, and so $u_{2d+1} \in V(F_1)$. If we continue this process we finally arrive at $u_{kd+1} = u_1$. This leads to $k = \frac{\mathrm{lcm}(n,d)}{d}$ and thus $|V(F_1)| = \frac{\mathrm{lcm}(n,d)}{d}$.

Furthermore, the definition of the circulant graph implies that $u_{p-d} \in V(F_1)$ for every vertex $u_p \in V(F_1)$. However, for reason of symmetry these vertices were already taken up before.

For $r > 1$, we also have $|V(F_1)| = \frac{\mathrm{lcm}(n,d)}{d}$, but the number of edges could increase. In both cases the component F_1 is isomorphic to $C_{\frac{\mathrm{lcm}(n,d)}{d}}(\{1,2,\dots,r\})$.

Possible further components $F_2, F_3, \dots, F_t$ of G are isomorphic to F_1. Applying Theorem 2.1, we obtain the desired result as follows:

$$\begin{aligned} & d_S(G) = 2r+1 \\ \iff & d_S(F_i) = 2r+1 \quad \text{for all components } F_i,\ i \in \{1,2,\dots,t\} \\ \iff & \frac{\mathrm{lcm}(n,d)}{d} \equiv 0 \pmod{2r+1} \ \square \end{aligned}$$

3 The torus $C_p \times C_q$

The *cartesian product* $G = G_1 \times G_2$ of two vertex disjoint graphs G_1 and G_2 has $V(G) = V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2 v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1 v_1 \in E(G_1)$. The cartesian product of two cycles C_p and C_q is called a *torus*. The torus $C_p \times C_q$ is 4-regular. Using Theorem 1.1 and 1.2, we obtain $d_S(C_p \times C_q) \in \{1,3,5\}$. Theorem 3.3 conveys a characterisation of the torus graphs with $d_s(G) = 5$. For clarity, in the folling figures the vertices x with $f(x) = -1$ are colored black.

Theorem 3.1 If $q \geq 3$, then

$$d_S(C_3 \times C_q) \neq 5$$

Proof. Suppose on the contrary that $d_S(C_3 \times C_q) = 5$. Because of Theorem 1.3, there exists a signed dominating function f such that, without loss of generality, $f(x_{1,2}) = f(x_{2,2}) = -1$. This implies $f(x_{1,1}) = f(x_{2,1}) = f(x_{3,2}) = f(x_{1,3}) = f(x_{2,3}) = 1$, and thus it follows that $f(x_{3,1}) = f(x_{3,3}) = 1$ (see Figure 1).

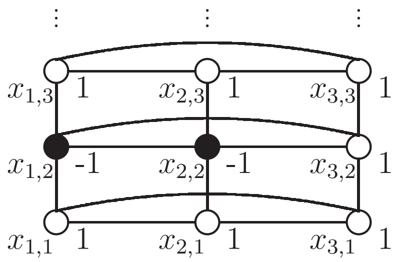

Figure 1

Hence $\sum_{y \in N[x_{3,3}]} f(y) \geq 3$, a contradiction to Theorem 1.3. □

Theorem 3.2 If $q \geq 4$, then

$$d_S(C_4 \times C_q) \neq 5.$$

Proof. Suppose on the contrary that $d_S(C_4 \times C_q) = 5$. Let f be an arbitrary signed dominating function, and let $x_{2,2}$ be a vertex with $f(x_{2,2}) = -1$. Because of Theorem 1.3, there exists a vertex, say $x_{3,2}$, such that $f(x_{3,2}) = -1$. This implies $f(x_{1,2}) = f(x_{2,1}) = f(x_{3,1}) = f(x_{4,2}) = f(x_{3,3}) = f(x_{2,3}) = 1$ (see Figure 2).

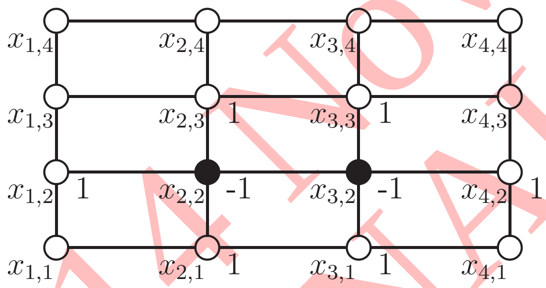

Figure 2

Now $x_{1,j}$ is adjacent to $x_{4,j}$ for $j \in \{1,2,3,4\}$ or $x_{i,1}$ is adjacent to $x_{i,4}$ for $i \in \{1,2,3,4\}$.

Case 1: Assume that $x_{1,j}$ is adjacent to $x_{4,j}$ for $j \in \{1,2,3,4\}$.

Subcase 1.1: Assume that $f(x_{1,1}) = -1$. This implies $f(x_{1,3}) = 1$. Hence, according to Theorem 1.3, we obtain $f(x_{1,4}) = f(x_{2,4}) = -1$. Now we conclude that $f(x_{3,4}) = f(x_{4,4}) = 1$. Hence $\sum_{y \in N[x_{4,3}]} f(y) \geq 3$, a contradiction to Theorem 1.3.

Subcase 1.2: Assume that $f(x_{1,1}) = 1$. This leads to $f(x_{4,1}) = -1$. Because of symmetry, the vertex $x_{4,1}$ can be looked upon as the vertex $x_{1,1}$ in Subcase 1.1.

Case 2: Assume that $x_{i,1}$ is adjacent to $x_{i,4}$ for $i \in \{1, 2, 3, 4\}$.

Subcase 2.1: Assume that $f(x_{1,1}) = -1$. This yields to $f(x_{1,3}) = f(x_{2,4}) = 1$ and thus $\sum_{y \in N[x_{2,3}]} f(y) = 3$, a contradiction to Theorem 1.3.

Subcase 2.2: Assume that $f(x_{1,1}) = 1$. It follows that $f(x_{1,4}) = f(x_{2,4}) = -1$. Hence we obtain $f(x_{3,4}) = f(x_{1,3}) = 1$. This implies $f(x_{4,3}) = -1$ and so $f(x_{4,1}) = 1$. We finally deduce that $\sum_{y \in N[x_{3,1}]} f(y) \geq 3$, a contradiction to Theorem 1.3. □

Theorem 3.3 If $p \geq 3$ and $q \geq 3$, then

$$d_S(C_p \times C_q) = 5 \iff p \equiv 0 \pmod 5 \wedge q \equiv 0 \pmod 5).$$

Proof. In view of Theorems 3.1 and 3.2, we assume in the following that $p \geq 5$ and $q \geq 5$. The case $p = q = 5$ will be investigated at the end of the proof, and hence we assume first that $p \geq 6$. We denote the vertices with $x_{i,j}$, where $j \in \{1, 2, \ldots, p\}$ and $i \in \{1, 2, \ldots, q\}$.

We assume that $d_S(C_p \times C_q) = 5$. Let f be an arbitrary signed domination function. First we show that f has a certain structure on a partial square with 5×5 vertices.

Because of Theorem 1.3, there exists a signed dominating function f such that, without loss of generality, $f(x_{2,2}) = f(x_{3,2}) = -1$. This implies $f(x_{1,2}) = f(x_{2,1}) = f(x_{3,2}) = f(x_{3,3}) = f(x_{2,4}) = f(x_{1,3}) = 1$ (see Figure 3).

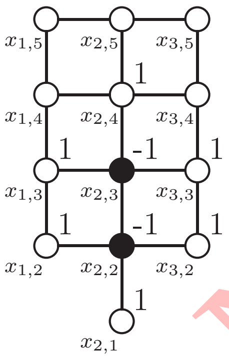

Figure 3

Applying Theorem 1.3, we observe that either $f(x_{2,5}) = -1$ or, without loss of generality, $f(x_{3,4}) = -1$.

Case 1: Assume that $f(x_{2,5}) = -1$. This leads to $f(x_{1,4}) = f(x_{3,4}) = 1$ and either $f(x_{3,5}) = 1$ or $f(x_{1,5}) = 1$, say $f(x_{3,5}) = 1$ (see Figure 4). Thus $\sum_{y \in N[x_{3,4}]} f(y) \geq 3$, a contradiction to Theorem 1.3.

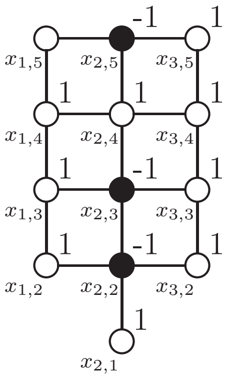

Figure 4

Case 2: Assume that $f(x_{3,4}) = -1$. This implies $f(x_{1,4}) = f(x_{2,5}) = 1$ and either $f(x_{4,4}) = -1$ or $f(x_{3,5}) = -1$.

Subcase 2.1: Assume that $f(x_{4,4}) = -1$. This yields to $f(x_{3,5}) = f(x_{4,5}) = f(x_{4,3}) = f(x_{5,4}) = 1$ (see Figure 5) and either $f(x_{5,5}) = 1$ or $f(x_{5,5}) = -1$.

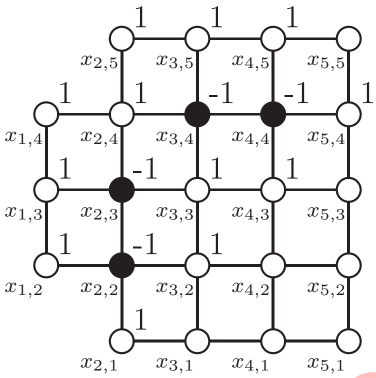

Figure 5

Subcase 2.1.1: Assume that $f(x_{5,5}) = 1$. This leads to $f(x_{4,6}) = f(x_{3,6}) = -1$ and so $f(x_{2,6}) = 1$ (see Figure 6). Thus $\sum_{y \in N[x_{2,5}]} f(y) \geq 3$, a contradiction to Theorem 1.3.

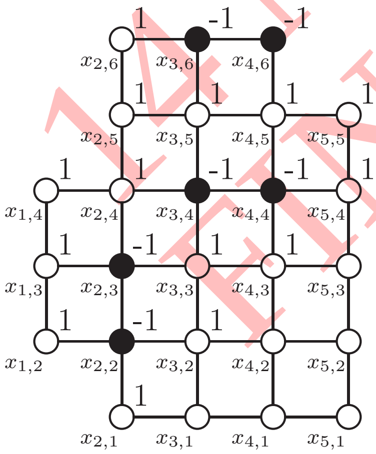

Figure 6

Subcase 2.1.2: Assume that $f(x_{5,5}) = -1$. Then we deduce that $f(x_{5,3}) = 1$ and thus $f(x_{4,2}) = f(x_{5,2}) = -1$. This leads to $f(x_{3,1}) = f(x_{4,1}) = f(x_{5,1}) = 1$ (see Figure 7). Hence $\sum_{y \in N[x_{3,1}]} f(y) \geq 3$, a contradiction to Theorem 1.3.

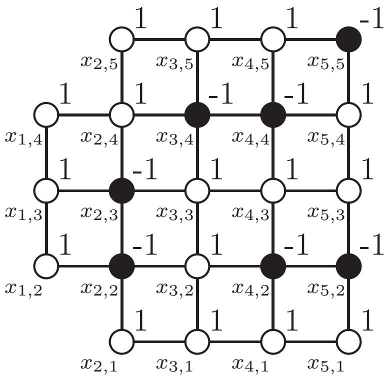

Figure 7

Subcase 2.2: Assume that $f(x_{3,5}) = -1$. Then we obtain $f(x_{4,4}) = f(x_{4,5}) = f(x_{3,6}) = 1$. It follws that $f(x_{4,3}) = 1$ and thus $f(x_{4,2}) = f(x_{5,3}) = -1$ and so $f(x_{3,1}) = 1$. If $x_{3,6}$ is adjacent to $x_{3,1}$, then we arrive at the contradiction $\sum_{y \in N[x_{3,1}]} f(y) \geq 3$. Therefore we deduce that that $f(x_{2,6}) = -1$ or $f(x_{4,6}) = -1$ or $f(x_{3,7}) = -1$. Since $f(x_{3,7}) = -1$ leads to the same situation as in Case 1, we investigate next the cases $f(x_{2,6}) = -1$ or $f(x_{4,6}) = -1$.

Subcase 2.2.1: Assume that $f(x_{2,6}) = -1$. This implies $f(x_{1,5}) = 1$ (see Figure 8), and we obtain the contradiction $\sum_{y \in N[x_{1,4}]} f(y) \geq 3$.

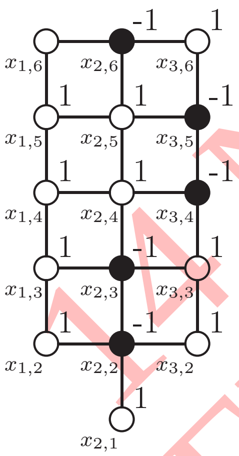

Figure 8

Subcase 2.2.2: Assume that $f(x_{4,6}) = -1$. This leads to $f(x_{2,6}) = 1$. It follows that $f(x_{5,4}) = -1$ and therefore $f(x_{5,2}) = f(x_{5,5}) = f(x_{6,3}) = f(x_{6,4}) = f(x_{6,2}) = 1$ (see Figure 9).

If $f(x_{5,6}) = -1$, then with the vertices $x_{3,4}$, $x_{3,5}$, $x_{4,6}$ and $x_{5,6}$, we arrive at the situation of Subcase 2.1. Hence we assume that $f(x_{5,6}) = 1$. This yields to $f(x_{6,5}) = -1$ (see Figure 9).

If $f(x_{6,6}) = 1$, then with the vertices $x_{5,3}$, $x_{5,4}$, $x_{6,5}$ and $x_{7,5}$, we have the situation of Subcase 2.1. Thus we now assume that $f(x_{6,6}) = -1$ (see Figure 9).

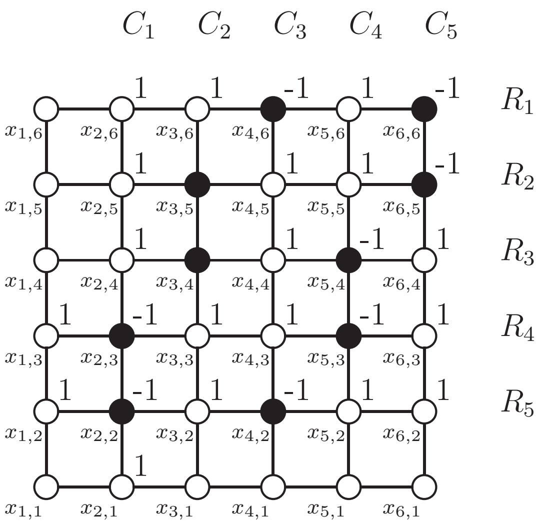

Figure 9

The vertices $x_{i,j}$ for $i, j \in \{2, 3, 4, 5, 6\}$ lead to a square with rows R_1, R_2, R_3, R_4, R_5 and columns C_1, C_2, C_3, C_4, C_5 with a fixed function f (see Figure 9). Now it is straightforward to verify that f is a signed dominating function only if $p \equiv 0 \pmod 5$ and $q \equiv 0 \pmod 5$.

The following five functions f_1, f_2, f_3, f_4, f_5 (see Figure 10) lead to a desired signed dominating family for $p = q = 5$.

For $p = 5k_1$ and $q = 5k_2$ (k_1 and k_2 arbitrary) we enumerate the vertices with $y_{i,j}$, $i \in \{1, 2, \ldots, p\}$, $j \in \{1, 2, \ldots, q\}$ provided that $y_{i,j}$ is adjacent to the vertices $y_{i-1,j}$, $y_{i+1,j}$, $y_{i,j-1}$ and $y_{i,j+1}$, where the indices are taken modulo p or modulo q, respectively. For $y_{i,j}$ we have $i = 5h_1 + i_0$ and $j = 5h_2 + j_0$ with $i_0, j_0 \in \{1, 2, 3, 4, 5\}$. If we define $g_k(y_{i,j}) = f_k(x_{i_0,j_0})$ for $k \in \{1, 2, 3, 4, 5\}$, then it is a simple matter to verify that each g_k is a signed dominating function. In this case we have $\sum_{i=1}^{5} g_i(y) = 1$ for every vertex $y \in V(C_p \times C_q)$, and thus $\{g_1, g_2, g_3, g_4, g_5\}$ is a signed dominating family. $\square$

If $p \equiv 0 \pmod 3$ and $q \geq 3$ is arbitrary, then we can show that $d_S(C_p \times C_q) = 3$.

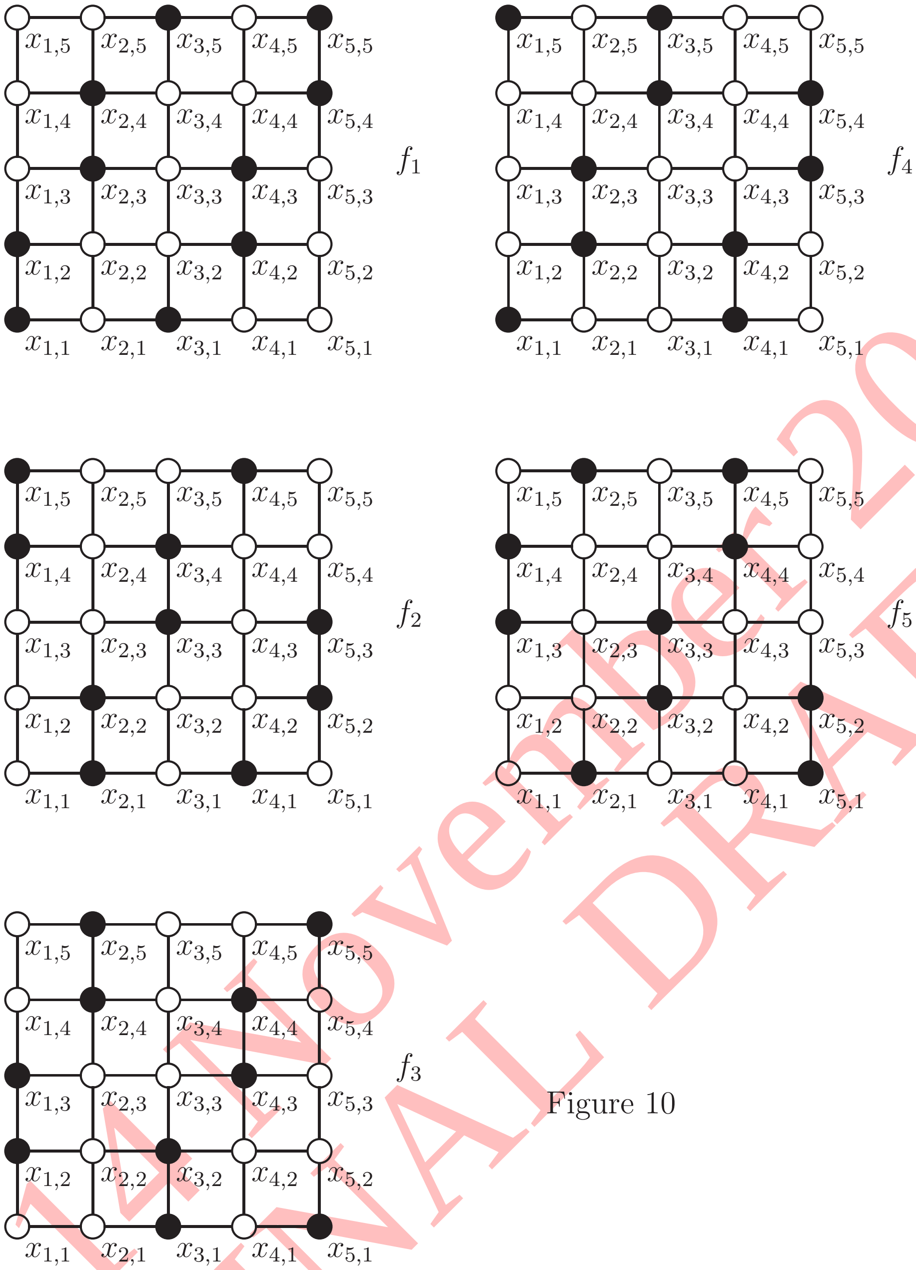

Figure 10

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