All 2-connected in-tournaments that are cycle complementary

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Abstract

An in-tournament is an oriented graph such that the negative neighborhood of every vertex induces a tournament. A digraph D is cycle complementary if there exist two vertex-disjoint directed cycles spanning the vertex set of D. Let D be a 2-connected in-tournament of order at least 8. In this paper we show that D is not cycle complementary if and only if it is 2-regular and has odd order.

Keywords: In-tournaments; complementary cycles.

1 Introduction

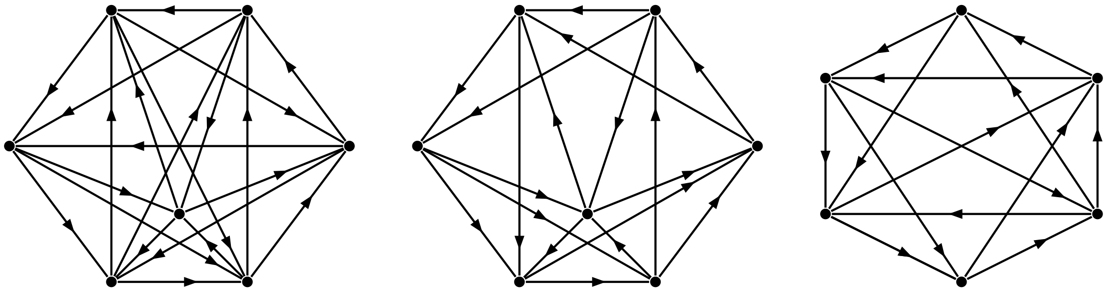
In 1990, Bang-Jensen [1] defined *local tournaments* to be the family of oriented graphs, i.e. digraphs without loops, multiple arcs and cycles of length 2, where the positive as well as the negative neighborhood of every vertex induces a tournament. In transfering the general adjacency only to vertices that have a common negative or a common positive neighbor, local tournaments form an interesting generalization of tournaments. Since then a lot of research has been done concerning local tournaments, or the more general class of *locally semicomplete digraphs*, where there might be cycles of length 2. In particular, the Ph.D. theses of Guo [11] and Huang [14] handeled this subject in detail. For more information concerning different generalizations of tournaments, the reader may be refered to the survey article of Bang-Jensen and Gutin [4]. In claiming adjacency only for vertices that have a common positive neighbor, local tournaments can be further generalized to the class of in-tournaments. An oriented graph D is called *in-tournament* if the set of negative neighbors of each vertex of D induces a tournament. Some problems concerning in-tournaments have been studied by Bang-Jensen, Huang and Prisner [6]. For information about the cycle structure of in-tournaments see, for example, Peters and Volkmann [16], Tewes [19], [20] or Tewes and Volkmann [21], [22].

Throughout this paper, *cycles* and *paths* are directed cycles and directed paths. Two subdigraphs of a digraph D are called *complementary* if they are disjoint and span the vertex set of D. A digraph is called *cycle complementary* if it has two complementary cycles. The general problem of partitioning a highly connected tournament into two subtournaments of high connectivity was mentioned by Thomassen (see Reid [17]). The first step towards the solution of this problem was made by Reid [17] in 1985 by the following result.

Theorem 1.1 (Reid [17] 1985). *Let T be a 2-connected tournament on $n \geq 6$ vertices. Then T contains two vertex-disjoint cycles of lengths 3 and $n-3$ unless T is isomorphic to T_7^1, where T_7^1 is the 3-regular tournament presented in Fig. 1.*

This result is stronger in the way that one of the strongly connected subtournaments can be specified to be a 3-cycle. For extensions, supplements and generalizations of Theorem 1.1 see, for example, Song [18], Guo and Volkmann [13], Bang-Jensen, Guo and Yeo [3], Chen, Gould and Li [9] and Gould and Guo [10].

An obvious necessary condition for a digraph D of order n to contain two complementary cycles is that the girth of D is at most $n/2$. In [2], Bang-Jensen observed that the second power C_{2k+1}^2 of an



T_7^1 T_7^2 T_6

Fig. 1: Three 2-connected local tournaments that are not cycle complementary.

odd cycle has girth $k+1$ and that the 2-regular digraph C^2_{2k+1} is a 2-connected local tournament. This shows that Theorem 1.1 cannot be extended to local tournaments in general. Confirming two conjectures by Bang-Jensen [2], Guo and Volkmann [12] proved that the second power of odd cycles are the only exceptions when $n \geq 8$.

Theorem 1.2 (Guo & Volkmann [12] 1994)**.** *Let D be a 2-connected local tournament on $n \geq 6$ vertices. Then D has two complementary cycles if and only if D is not the second power of an odd cycle and D is not a member of $\{T_7^1, T_7^2, T_6\}$, where T_7^1, T_7^2 and T_6 are presented in Fig. 1.*

In this paper we will show that Theorem 1.2 remains valid for the superclass of in-tournaments. The proof is much more difficult than the one of Theorem 1.2, since the structural properties of in-tournaments are not as strong as these of local tournaments.

2 Terminology and preliminary results

We assume that the reader is familiar with the basic concepts of graph theory and we refer to the comprehensive books by Bondy and Murty [7] or by Bang-Jensen and Gutin [5] for information which are not given here.

All digraphs mentioned in this paper are finite without loops and multiple arcs. For a digraph D we denote by $V(D)$ and $E(D)$ the *vertex set* and *arc set* of D, respectively. The subdigraph induced by a subset A of $V(D)$ is denoted by $D[A]$. A cycle with the vertices $x_1, x_2, \ldots, x_k$ and the arcs $x_1x_2, x_2x_3, \ldots, x_kx_1$ is called a *k-cycle* and is denoted by $x_1x_2 \ldots x_kx_1$. If we consider a k-cycle $C = x_1x_2 \ldots x_kx_1$ in a digraph D, all subscripts appearing in related calculations are taken modulo the cycle length k (note that $x_0 = x_k$). Let $C[x_i, x_j]$, where $1 \leq i, j \leq k$, denote the subpath $x_ix_{i+1} \ldots x_j$ of C with initial vertex x_i and terminal vertex x_j. If x is a vertex of C, the successor (predecessor) of x on C is denoted by x^+_C (x^-_C), and if no confusion arises, x^+ and x^- will be used instead of x^+_C and x^-_C, respectively. The notations for paths are defined analogously.

If $xy \in E(D)$, we say that x *dominates* y. If A and B are two disjoint subdigraphs of a digraph D such that every vertex of A dominates every vertex of B, we say that A *dominates* B, denoted by $A \rightarrow B$. Furthermore, $A \rightsquigarrow B$ denotes the fact that there is no arc leading from B to A and at least one arc leading from A to B. In this case we also say that A *weakly dominates* B. The *outset* (*inset*) $N^+(x)$ ($N^-(x)$) of a vertex x is the set of positive (negative) neighbors of x. More generally, for arbitrary subdigraphs A and B of D, the outset $N^+(A, B)$ is the set of vertices in B to which there is an arc from a vertex in A, and the inset $N^-(A, B)$ is defined analogously. The numbers $|N^+(x)|$ and $|N^-(x)|$ are called *outdegree* and *indegree* of x, respectively. We say that a digraph D is *k-regular* if $|N^+(x)| = |N^-(x)| = k$ for every vertex x of D.

If D is a strong digraph and S is a subset of $V(D)$ such that $D - S$ is not strong, we say that S is a *separating set*. A separating set S is called *minimal separating set* (*minimum separating set*) if there exists no separating set U such that $U \subseteq S$ and $U \neq S$ ($|U| < |S|$).

The first result is a simple, but powerful observation on the interaction of a cycle and an external vertex.

Lemma 2.1. *Let D be an in-tournament containing a cycle $C = u_1 u_2 \dots u_t u_1$.*

(a) If there exists a vertex $x \in V(D) - V(C)$ such that $d^+(x, C) > 0$, either $x \to C$ or $u_i \to x \to u_{i+1}$ for some $1 \leq i \leq t$.

(b) If $P = v_1 v_2 \dots v_s$ is a path in $D - V(C)$ such that $d^+(v_s, C) > 0$, either there exists an integer $1 \leq i \leq s$ such that $v_i \to C$, $v_i \to P[v_{i+1}, v_s]$ and D has a cycle that consists of all vertices of C and $P[v_{i+1}, v_s]$ or D contains a Hamiltonian cycle of $D[V(C) \cup V(P)]$.

Proof. (a) Without loss of generality, let $x \to u_t$. Assume that x does not dominate C. Obviously, x and u_{t-1} are negative neighbors of the vertex u_t and hence, since D is an in-tournament, they are adjacent. If $u_{t-1} \to x$, we choose $i = t - 1$ and are done. Otherwise $x \to u_{t-1}$ which implies the adjacency of the vertices u_{t-2} and x. Since x does not dominate C, we obtain i in at most $t - 1$ steps.

(b) Using the first part of this lemma, we conclude that either $v_s \to C$ or there exists an integer $1 \leq j \leq t$ such that $u_j \to v_s \to u_{j+1}$. If $v_s \to C$, we choose $i = s$ and are done. Otherwise note that we can extend the cycle C by the vertex v_s to a cycle C' and that $d^+(v_{s-1}, C') > 0$. Using these observations we obtain i in at most s steps. □

Camion [8] proved in 1959 that a tournament is Hamiltonian if and only if it is strong. In 1993, Bang-Jensen, Huang and Prisner [6] extended this result to in-tournaments.

Theorem 2.2 (Bang-Jensen, Huang & Prisner [6] 1993)**.** *An in-tournament is Hamiltonian if and only if it is strong.*

The previous results are useful for the analyzation of the structural properties of in-tournaments.

Theorem 2.3 (Bang-Jensen, Huang & Prisner [6] 1993)**.** *Let D be a strong in-tournament and let S be a minimal separating set of D.*

(a) If A and B are two distinct strong components of $D - S$, either there is no arc between them or A weakly dominates B or B weakly dominates A. Furthermore, if A weakly dominates B, the set $N^-(B, A)$ dominates B.

(b) If A and B are two distinct strong components of $D - S$ such that A weakly dominates B, the set $N^-(b, A)$ induces a tournament for each $b \in B$.

(c) The strong components of $D - S$ can be ordered in a unique way $D_1, D_2, \dots, D_p$ such that there are no arcs from D_j to D_i for $j > i$, and D_i has an arc to D_{i+1} for $i = 1, 2, \dots, p - 1$.

According to Theorem 2.3, we give the following definition.

Definition 2.4. *The unique labelling $D_1, D_2, \dots, D_p$ of the strong components of $D - S$ as described in Theorem 2.3 is called* the strong decomposition of $D - S$*. We call D_1 the* initial *and D_p the* terminal component.

The following results are immediate by Theorem 2.3.

Corollary 2.5 (Bang-Jensen, Huang & Prisner [6] 1993)**.** *Let D be a strong in-tournament and let S be a minimal separating set of D. The strong decomposition of $D - S$ has the following properties.*

(a) *If* $x_i \to x_k$ *for* $x_i \in V(D_i)$ *and* $x_k \in V(D_k)$ *with* $1 \leq i \neq k \leq p$*, then* $x_i \to D_j$ *for every* $i+1 \leq j \leq k$*.*

(b) *The digraph* $D - S$ *has a Hamiltonian path.*

(c) *For every* $s \in S$ *we have* $d^+(s, D_1) > 0$ *and* $d^-(s, D_p) > 0$*.*

From the fact that every connected non-strong in-tournament has a unique strong decomposition, we can find a further useful decomposition. This result plays an important role in our proof.

Theorem 2.6 (Structure Theorem)**.** *Let* D *be a strong in-tournament and let* S *be a minimal separating set of* D*. There is a unique order* $D_1^{'}, D_2^{'}, \ldots, D_r^{'}$ *with* $r \geq 2$ *of the strong components of* $D - S$ *such that*

(a) D'_1 *is the terminal component of* $D - S$ *and* D'_i *consists of some strong components of* D *for* $i \geq 2$*;*

(b) *there exists a vertex* x *in the initial component of* D'_{i+1} *and a vertex* y *in the terminal component of* D'_{i+1} *such that* $\{x, y\}$ *dominates the initial component of* D'_i *for* $i = 1, 2, \ldots, r-1$*;*

(c) *there are no arcs between* D'_i *and* D'_j *for* i, j *satisfying* $|i - j| \geq 2$*;*

(d) *if* $r \geq 3$*, there exist no arcs from* D'_i *to* S *for* $i \geq 3$*,* $S \to D_1$ *and* S *induces a tournament in* D*.*

Proof. Let $D_1, D_2, \ldots, D_p$ be the strong decompostion of $D - S$. We define (see Fig. 2)

$$D_1^{'} = D_p, \quad \lambda_1 = p,$$

$$\lambda_{i+1} = \min\left\{ j \mid N^+(D_j, D_i^{'}) \neq \emptyset \right\}$$

and

$$D_{i+1}^{'} = D\left[V(D_{\lambda_{i+1}}) \cup V(D_{\lambda_{i+1}+1}) \cup \ldots \cup V(D_{\lambda_i - 1})\right].$$

So we have a new decomposition $D'_1, D'_2, \ldots, D'_r$, where $2 \leq r \leq p$, of D that satisfies (a).

By the definition of D'_{i+1}, there exists a strong component D_l of D'_i such that $N^+(D_{\lambda_{i+1}}, D_l) \neq \emptyset$. Therefore we conclude from Corollary 2.5 (a) that there exists a vertex $x \in V(D_{\lambda_{i+1}})$ such that $x \to D_j$ for each $j \in \{\lambda_{i+1}, \ldots, l\}$. From Theorem 2.3 (c) and Corollary 2.5, it follows that there exists a vertex $y \in V(D_{\lambda_i - 1})$ such that $y \to D_{\lambda_i}$. So (b) has been proved.

Note that if $r = 2$, there is nothing to prove in (c). If $r \geq 3$ and i, j are two integers with $i \geq j+2$, there is no arc from D'_i to D'_j by the definition of λ_{i-1}. In addition, D contains no arc from D'_j to D'_i by Theorem 2.3 (c).

Assume to the contrary that there is an arc xs from $x \in V(D'_i)$ to $s \in S$, where $i \geq 3$. Note that Corollary 2.5 (c) states that s has a negative neighbor x' in D_p. Since D is an in-tournament, it follows that x and x' are adjacent, a contradiction to (c).

Now we shall prove that $S \to D_1$. Note that we have $d^+(s, D_1) > 0$ for every vertex $s \in S$ by Corollary 2.5 (b). Now let $s \in S$ be an arbitrary vertex. If D_1 consists of a single vertex, there is nothing to prove. Otherwise D_1 has a Hamiltonian cycle by Theorem 2.2. Using Lemma 2.1 (a), we deduce that either $s \to D_1$ or that s has a negative neighbor in D_1. Thus, if $s \not\to D_1$, the vertex s has negative neighbors both in D_1 and D_p, a contradiction to (c). This completes the proof of this theorem. □

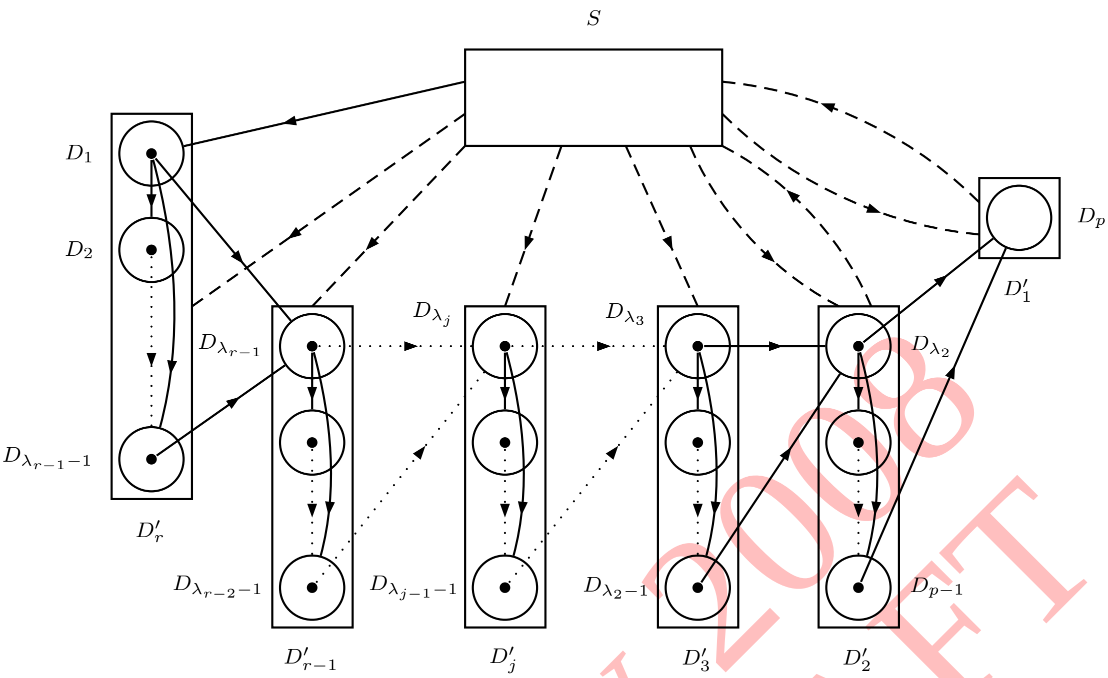


Fig. 2: The decomposition of a strong in-tournament.

3 Main Results

In this paper we shall give the following complete characterization of 2-connected in-tournaments which are cycle complementary.

Theorem 3.1 (Main Theorem). *Let D be a 2-connected in-tournament on $n \geq 6$ vertices that is not a member of $\{T_7^1, T_7^2, T_6\}$ as presented in Fig. 1. Then D is not cycle complementary if and only if D is 2-regular and $|V(D)|$ is odd.*

Proof of Main Theorem

We shall prove Theorem 3.1 for $n \geq 8$. For $n = 6$ and $n = 7$ it is straightforward to verify the desired result by means of a case by case analysis.

Suppose that D is k-connected, but not $(k+1)$-connected ($k \geq 2$). Then D has a separating set S of size k. According to Corollary 2.5 (b) and Theorem 2.6, the digraph $D - S$ is connected and we have a new order $D_1^{'}, D_2^{'}, \ldots, D_r^{'}$, where $2 \leq r \leq p$, of the strong components $D_1, D_2, \ldots, D_p$ of $D - S$ such that there are only arcs from $D_{i+1}^{'}$ to $D_i^{'}$ for $i = 1, 2, \ldots, r-1$.

Note that the k-connectivity of D implies that each subdigraph $D_i^{'}$, where $2 \leq i \leq r-1$, contains at least k vertices. Furthermore, we may assume, without loss of generality, that every vertex of $S - s_1$ has at least two positive neighbors in $D - S$.

Claim. *If $\sum_{j=1}^{\lambda_2 - 1} |V(D_j)| \geq 2$, we have $|V(D_i)| = 1$ for each $i \leq \lambda_2$.*

Proof. Assume that $|V(D_i)| \geq 3$ for an index $i \leq \lambda_2$. Let

$$A := \bigcup_{j=1}^{\lambda_2 - 1} V(D_j).$$

Since D is 2-connected, we have $|N^-(D_i, A)| \geq 2$ which implies that D contains two distinct vertices $v_1, v_2 \in A$ that dominate D_i. By a well-known result due to Menger [15] and Whitney

[23], we obtain two vertex-disjoint paths leading from D_i to $\{v_1, v_2\}$ and therefore, by adding the appropriate arcs from $\{v_1, v_2\}$ to D_i, two vertex-disjoint cycles C_1, C_2 in D. We choose C_1 and C_2 such that $|V(C_1) \cup V(C_2)|$ is maximal. We will now show that $V(C_1) \cup V(C_2) = V(D)$ which is a contradiction to our assumption that D is not cycle complementary.

Let $u \notin V(C_1) \cup V(C_2)$ be an arbitrary vertex that has a positive neighbor in $V(C_1) \cup V(C_2)$, say $N^+(u, C_1) \neq \emptyset$. By Lemma 2.1 and the maximality of the cycles, it follows that $u \to C_1$. Note that each of the two cycles contains at least one vertex of A, one vertex of D_i and one vertex of S. This implies that u has positive neighbors both in D_i and S. With the help of Theorem 2.6 we conclude that $u \in S$.

By the observations above we conclude that $V(D) - S \subseteq V(C_1) \cup V(C_2)$. Note that each vertex $s \in S$ dominates D_1 by Theorem 2.6. It follows that each vertex $s \in S$ has a positive neighbor on C_1 or C_2. In addition, if $s \in S - (V(C_1) \cup V(C_2))$ has a positive neighbor on C_j, where $j \in \{1, 2\}$, the vertex s dominstes C_j and thus, $N^+(s, D_i) \neq \emptyset$. It follows that $s \to A$ by Theorem 2.6. The latter implies that s has positive neighbors on both cycles. Since C_1 and C_2 were chosen maximal, we conclude that $s \to C_1$ and $s \to C_2$ and thus, $s \to D - S$, a contradiction to Corollary 2.5. This completes the proof of this claim. $\square$

Suppose that D is not cycle complementary. We shall show below that then D is 2-regular and $|V(D)|$ is odd. We consider two cases, depending on the value of r.

Case 1: Let $r \geq 3$. By Theorem 2.6, there exist no arcs from D'_i to S for $i \geq 3$, $S \to D_1$ and S induces a tournament. In addition, if $k \geq 4$, the tournament $D[S]$ is transitive, since otherwise an arbitrary 3-cycle C_3 of $D[S]$ and a Hamiltonian cycle of $D - C_3$ are complementary cycles of D. Let $s_1 s_2 \ldots s_k$ be a Hamiltonian path of $D[S]$. Note that s_k has at least two positive neighbors outside of S. By the claim above we have $|V(D_i)| = 1$ for each $i \leq \lambda_2$.

Note that for $3 \leq j \leq r$ there exists an unique Hamiltonian path $x_1^j x_2^j \ldots x_{n_j}^j$ of D'_j such that $x_1^j \to x_l^j$ for each $l > 1$. In addition, if $x_1^j x_2^j \ldots x_{n_j}^j$ is a Hamiltonian path of D'_j and $x_1^{j-1} x_2^{j-1} \ldots x_{n_{j-1}}^{j-1}$ is a Hamiltonian path of D'_{j-1}, where $j \geq 2$, the vertex x_1^j dominates $D_{\lambda_{j-1}}$ and $x_{n_j}^j$ dominates x_2^{j-1}.

Subcase 1.1: Suppose that $|V(D_p)| \geq 3$. Let C be a Hamiltonian cycle of D_p and let $z_1, z_2 \in V(D_p)$ be two vertices such that $z_1 \to s_1$ and $z_2 \to s_k$. Then

$$s_{k-1} x_1^r x_1^{r-1} \ldots x_1^2 C[z_2^+, z_1] s_1 s_2 \ldots s_{k-1}$$

and

$$s_k x_2^r x_3^r \ldots x_{n_r}^r x_2^{r-1} x_3^{r-1} \ldots x_{n_{r-1}}^{r-1} \ldots x_2^2 x_3^2 \ldots x_{n_2}^2 C[z_1^+, z_2] s_k$$

are complementary cycles in D.

Subcase 1.2: Suppose that $|V(D_p)| = 1$. Note that in this case $N^+(D_{p-1}, S) \neq \emptyset$. Let $v \in V(D_{p-1})$ be a vertex that has a positive neighbor in S. Then either $v \to s_i$ for an index $i \neq k$ or $s_{k-1} \to v \to s_k$. In the latter case s_{k-1} has a negative neighbor $u \neq v$ in D'_2.

Subcase 1.2.1: Suppose that $|V(D'_r)| \geq 2$. Then $s_k \to x_2^r$ and $s_{k-1} \to x_1^r$.

Subcase 1.2.1.1: Suppose that $|V(D'_j)| \geq 3$ for an index $2 \leq j \leq r$. If $x_{n_2}^2 \to s_i$, where $i \neq k$, the cycles

$$C_1 = s_{k-1} x_1^r x_1^{r-1} \ldots x_1^j x_{n_j}^j x_2^{j-1} x_3^{j-1} \ldots x_{n_{j-1}}^{j-1} \ldots x_2^2 x_3^2 \ldots x_{n_2}^2 s_i s_{i+1} \ldots s_{k-1}$$

and

$$C_2 = s_k x_2^r x_3^r \ldots x_{n_r}^r \ldots x_2^{j+1} x_3^{j+1} \ldots x_{n_{j+1}}^{j+1} x_2^j x_3^j \ldots x_{n_j-1}^j x_1^{j-1} x_1^{j-2} \ldots x_1^1 s_k$$

are vertex-disjoint. If $i = 1$, the cycles C_1 and C_2 are complementary in D. If $i \geq 2$ and $D[S]$ is transitive, the path $s_1 s_2 \ldots s_{i-1}$ can be inserted in C_2. Otherwise we have $k = 3$, $i = 2$ and $D[S]$

induces the 3-cycle $s_1s_2s_3s_1$ in D. If $s_1 \not\to C_1$, the vertex s_1 can be inserted in C_1. Otherwise $s_1 \to C_1$ and it follows that $s_1 \to x_2^r$. But then

$$s_2s_3x_1^rx_1^{r-1}\dots x_1^jx_{n_j}^jx_2^{j-1}x_3^{j-1}\dots x_{n_{j-1}}^{j-1}\dots x_2^2x_3^2\dots x_{n_2}^2s_2$$

and

$$s_1x_2^rx_3^r\dots x_{n_r}^r\dots x_2^{j+1}x_3^{j+1}\dots x_{n_{j+1}}^{j+1}x_2^jx_3^j\dots x_{n_j-1}^jx_1^{j-1}x_1^{j-2}\dots x_1^1s_1$$

are complementary cycles of D.

If there exists no arc $x_{n_2}^2s_i$ in D such that $i \neq k$, we obtain $s_{k-1} \to x_{n_2}^2 \to s_k$. In this case

$$s_{k-1}x_1^rx_1^{r-1}\dots x_1^1s_1s_2\dots s_{k-1} \quad \text{and} \quad s_kx_2^rx_3^r\dots x_{n_r}^r\dots x_2^2x_3^2\dots x_{n_2}^2s_k$$

show that D is cycle complementary.

Subcase 1.2.1.2: Suppose that $D_j^{'}$ is a 1-path for each $2 \leq j \leq r$. Note that we have $k = 2$ in this case. If D is not 2-regular, at least one of the following possibilities holds. The digraph D has an arc

(i) s_1z, where $z \in V(D) - \{s_2, x_1^r, x_2^2, x_1^1\}$ or

(ii) s_2z, where $z \in V(D) - \{x_1^r, x_2^r, x_1^1, s_1\}$ or

(iii) $x_1^jx_2^{j-1}$, where $j \in \{3, 4, \dots, r\}$ or

(iv) x_1^2s, where $s \in S$ or

(v) $x_2^2s_2$.

But each such arc yields a contradiction to the fact that D is not cycle complementary which means that D is 2-regular.

Subcase 1.2.2: Suppose that $|V(D_r^{'})| = 1$.

Subcase 1.2.2.1: Suppose that $r \geq 4$. Then $s_k \to \{x_1^r, x_1^{r-1}\}$ and $s_{k-1} \to x_1^r$.

Subcase 1.2.2.1.1: Suppose that $|V(D_j^{'})| \geq 3$ for an index $2 \leq j \leq r-1$. If $x_{n_2}^2 \to s_i$, where $i \neq k$, the cycles

$$C_1 = s_{k-1}x_1^rx_2^{r-1}x_3^{r-1}\dots x_{n_{r-1}}^{r-1}\dots x_2^2x_3^2\dots x_{n_2}^2s_is_{i+1}\dots s_{k-1} \quad \text{and} \quad C_2 = s_kx_1^{r-1}x_1^{r-2}\dots x_1^1s_k$$

are vertex-disjoint. If $i = 1$, the cycles C_1 and C_2 are complementary in D. If $i \geq 2$ and $D[S]$ is transitive, the path $s_1s_2\dots s_{i-1}$ can be inserted in C_2. Otherwise we have $k = 3$, $i = 2$ and $D[S]$ induces the 3-cycle $s_1s_2s_3s_1$ in D. If $s_1 \not\to C_1$, the vertex s_1 can be inserted in C_1. Otherwise $s_1 \to C_1$ and it follows that $s_1 \to x_1^{r-1}$. But then

$$s_2x_1^rx_2^{r-1}x_3^{r-1}\dots x_{n_{r-1}}^{r-1}\dots x_2^2x_3^2\dots x_{n_2}^2s_2 \quad \text{and} \quad s_3s_1x_1^{r-1}x_1^{r-2}\dots x_1^1s_3$$

are complementary cycles of D.

If there exists no arc $x_{n_2}^2s_i$ in D such that $i \neq k$, then we obtain $s_{k-1} \to x_{n_2}^2 \to s_k$. In this case

$$s_{k-1}x_1^rx_2^{r-1}x_3^{r-1}\dots x_{n_{r-1}}^{r-1}\dots x_2^{j+1}x_3^{j+1}\dots x_{n_{j+1}}^{j+1}x_2^jx_3^j\dots x_{n_j-1}^jx_1^{j-1}\dots x_1^1s_1s_2\dots s_{k-1}$$

and

$$s_kx_1^{r-1}x_1^{r-2}\dots x_1^jx_{n_j}^jx_2^{j-1}x_3^{j-1}\dots x_{n_{j-1}}^{j-1}\dots x_2^2x_3^2\dots x_{n_2}^2s_k$$

show that D is cycle complementary.

Subcase 1.2.2.1.2: Suppose that $D_j^{'}$ is a 1-path for each $2 \leq j \leq r-1$. Note that we have $k = 2$ in this case. We consider the vertex x_2^2.

If $x_2^2 \to s_1$, the cycles

$$s_1 x_1^r x_2^{r-1} x_2^{r-2} \ldots x_2^2 s_1 \quad \text{and} \quad s_2 x_1^{r-1} x_1^{r-2} \ldots x_1^1 s_1$$

are complementary in D.

Otherwise $s_1 \to x_2^2 \to s_2$. By Theorem 2.6 it follows that $s_1 \to D - \{x_1^1, x_2^2\}$. Therefore

$$s_1 x_1^{r-1} x_1^{r-2} \ldots x_1^1 s_1 \quad \text{and} \quad s_2 x_1^r x_2^{r-1} x_2^{r-2} \ldots x_2^2 s_2$$

show that D is cycle complementary.

Subcase 1.2.2.2: Suppose that $r = 3$. Note that D_2 has at least $k-1$ negative neighbors in S and D_{p-1} has at least $k-1$ positive neighbors in S.

Subcase 1.2.2.2.1: Suppose that $p \geq 4$.

If there exists a vertex $s \in S$ such that $s \in N^+(D_{p-1}) \cap N^-(D_2)$, it is easy to see that there exists a subset A of D_2 (where $A \neq V(D_2)$ if $|V(D_2)| \geq 3$) and a subset B of D_{p-1} (where $B \neq V(D_{p-1})$ if $|V(D_{p-1})| \geq 3$) such that the digraphs

$$H := D[\{s\} \cup A \cup B] \quad \text{and} \quad D - V(H)$$

are both strong.

If $N^+(D_{p-1}) \cap N^-(D_2) = \emptyset$, we conclude that $k = 2$. We may assume, without loss of generality, that $s_1 \to s_2$ and $N^+(s_1, D_2) = N^-(s_2, D_{p-1}) = \emptyset$.

If $s_2 \to D_2$, there exists a subset X of D_2 (where $X \neq V(D_2)$ if $|V(D_2)| \geq 3$) such that

$$H := D[\{s_2, x_1^1\} \cup X] \quad \text{and} \quad D - V(H)$$

are both strong.

If $s_2 \not\to D_2$, there exists a subset A of D_2 (where $A \neq V(D_2)$ if $|V(D_2)| \geq 3$) and a subset B of D_{p-1} (where $B \neq V(D_{p-1})$ if $|V(D_{p-1})| \geq 3$) such that the digraphs

$$H := D[\{s_1, x_1^3\} \cup A \cup B] \quad \text{and} \quad D - V(H)$$

are both strong.

Subcase 1.2.2.2.2: Suppose that $p = 3$.

If $|V(D_2)| \geq 4$, we consider the positive and the negative neighborhood of D_2. The assumption that there exists a vertex $s \in S$ such that $N^+(D_2, S) = S - s = N^-(D_2, S)$ leads to a contradiction, since D is a k-connected in-tournament with $k \geq 2$. Thus, S contains distinct vertices $s_1 \neq s_2$ such that $S - s_2 \subseteq N^+(D_2)$, $S - s_1 \subseteq N^-(D_2)$ and $s_1 \to s_2$. Let C be a Hamiltonian cycle of D_2.

If $s_2 \to D_2$, note that there are two distinct vertices $z_1 \neq z_2$ in D_2 such that $z_1 \to s_1$ and $z_2 \to x_1^1$. But then

$$s_2 C[z_1^+, z_2] x_1^1 s_2 \quad \text{and} \quad s_1 x_1^3 C[z_2^+, z_1] s_1$$

are complementary cycles of D.

If $s_2 \not\to D_2$, there exists a vertex $z_2 \in V(D_2)$ such that $z_2 \to s_2 \to z_2^+$. Thus s_1 and z_2 are adjacent.

If $z_2 \to s_1$, note that there exists a vertex $z_1 \neq z_2$ in D_2 such that $z_1 \to x_1^1$. Now

$$s_2 C[z_2^+, z_1] x_1^1 \quad \text{and} \quad s_1 x_1^3 C[z_1^+, z_2]$$

are complementary cycles of D.

If $s_1 \to z_2$, there exists a vertex z_1 in D_2 such that $z_1 \to s_1 \to z_1^+$.

If $z_1^+ \neq z_2$, we consider the vertex set $A := V(C[z_1^+, z_2^-])$. Since D is strong, we have $N^+(A) - A \neq \emptyset$. If $A \ni a \to s_2$, the cycles

$$s_2 x_1^3 C[z_1^+, a] s_2 \quad \text{and} \quad s_1 C[a^+, z_1] x_1^1 s_1$$

are complementary in D. If $A \ni a \to z_1^1$, the cycles

$$s_2 x_1^3 C[z_1^+, a] x_1^1 s_2 \quad \text{and} \quad s_1 C[a^+, z_1] s_1$$

show that D is cycle complementary. Finally, if $A \ni a \to b \in V(D_2) - A$, the cycles

$$C_1 = s_1 C[z_1^+, a] C[b, z_1] x_1^1 s_1 \quad \text{and} \quad C_2 = s_2 x_1^3 C[a^+, z_2] s_2$$

are vertex-disjoint. By Lemma 2.1 each vertex v of $C[z_2^+, b^-]$ can either be inserted in C_1 or dominates C_1. Obviously D is cycle complementary in the first case. In the latter case let v be a vertex of $C[z_2^+, b^-]$ such that $v \to V(C_1) \cup V(C[v^+, b^-])$ and $|V(C[v^+, b^-])|$ is minimal. But then the sets

$$\{s_2, x_1^3, x_1^1\} \cup V(C[a^+, v]) \quad \text{and} \quad \{s_1\} \cup V(C[v^+, a])$$

both induce strong in-tournaments in D. Theorem 2.2 implies that D is cycle complementary.

If $z_1^+ = z_2$, we shall show in the first step that $N^-(s_1) = \{z_1, x_1^1\}$ and $N^+(s_1) = \{s_2, x_1^3, z_1^+\}$. If $v \neq z_1$ is a negative neighbor of s_1 in D_2, the cycles

$$s_1 C[z_1^+, v] s_1 \quad \text{and} \quad s_2 x_1^3 C[v^+, z_1] x_1^1$$

are complementary in D. We may assume now that $N^-(s_1, D_2) = \{z_1\}$. If $w \neq z_1^+$ is a positive neighbor of s_1 in D_2, our assumption implies that $s_1 \to z_2^+$ and thus,

$$s_1 C[z_2^+, z_1] s_1 \quad \text{and} \quad s_2 x_1^3 z_2 x_1^1 s_2$$

are complementary cycles of D. Now note that $N^+(s_1) = \{s_2, x_1^3, z_1^+\}$ is a minimal separating set of D. Let $A_1, A_2, \ldots, A_q$ be the strong decomposition and $A'_1, A'_2, \ldots, A'_t$ be the decomposition according to Theorem 2.6 of $D - \{s_2, x_1^3, z_1^+\}$. Then $V(A_q) = \{s_1\}$, $V(A_{q-1}) = \{x_1^1\}$ and $q \geq 4$. If $z_1 \notin V(A_1)$, we obtain $t \geq 3$ and $q \geq 4$. This case is already solved. Otherwise $z_1 \in V(A_1)$ and $t = 2$. Note that $x_1^3 \to A'_1$ and that z_1^+ has a positive neighbor in A_1 by Theorem 2.6. Therefore both

$$\{s_1, z_1^+\} \cup V(A_1) \quad \text{and} \quad \{s_2, x_1^3, x_1^1\} \cup \bigcup_{i=2}^{q-2} V(A_i)$$

induce strong in-tournaments in D and thus, D is cycle complementary by Theorem 2.2.

If $|V(D_2)| = 3$, we obtain $k = 3$ and $|V(D)| = 8$. Since $|N^+(S)|, |N^+(D_2)| \geq k = 3$, there exist two non-incident arcs leading from D_2 to S and two non-incident arcs leading from S to D_2. Now it is easy to check that D is cycle complementary.

Case 2: Let $r = 2$ (see Fig. 3). Note that $N^-(D_1) = S = N^+(D_p)$ and $d^+(s_i, D_1), d^-(s_i, D_p) \geq 1$ for every $i \in \{1, 2, \ldots, k\}$ (see Corollary 2.5 (c)). If we consider a strong component D_i, all predecessors and successors refer to the corresponding Hamiltonian cycle of D_i, unless stated otherwise. Furthermore, we may assume that $r = 2$ for any separating set S of size k. Now we consider three subcases depending on the value of k.

Subcase 2.1: Suppose that $k \geq 4$. Note that D_1 contains a vertex that dominates D_p and that every vertex $s \in S$ has at least one negative neighbor in D_p. It follows that if $|V(D_1)| = 1$ or S contains a vertex that dominates D_1, the digraph D has a 3-cycle C. Since D is at least 4-connected, the remaining digraph $D - V(C)$ is strong and hence, in view of Theorem 2.2, Hamiltonian. It follows that D is cycle complementary.

Therefore we may assume that $|V(D_1)| \geq 3$ and that for every vertex $s_i \in S$, there exists a vertex $y_i \in V(D_1)$ such that $y_i \to s_i \to y_i^+$.

Subcase 2.1.1: Suppose that $|V(D_p)| \geq 3$.

Assume that $p = 2$. Since each vertex of S has a positive as well as a negative neighbor in D_1, it is possible to insert every vertex of S in a Hamiltonian cycle of D_1. This extended cycle and a Hamiltonian cycle of D_2 are complementary cycles of D.

Therefore we may now assume that $p \geq 3$. Let C be a Hamiltonian cycle of D_1. If, without loss of generality, $y_1 \neq y_2$, there exist complementary paths P_1 and P_2 of D_p such that the terminal vertex of P_1 dominates s_1 and the terminal vertex of P_2 dominates s_2. It follows that

$$C_1 = s_1 C[y_1^+, y_2] P_1 s_1 \quad \text{and} \quad C_2 = s_2 C[y_2^+, y_1] D_2 D_3 \dots D_{p-1} P_2 s_2$$

are vertex-disjoint cycles in D. We show next that all vertices s_m, where $m \geq 3$, can be inserted in at least one of these cycles. Note that the vertex s_m has a positive neighbor $y \in V(D_1)$. If, without loss of generality, $y \in V(C_1)$, the vertex s_m can be inserted in C_1 unless $s_m \to C_1$. Let P_2 and P_m be complementary paths of D_p such that the terminal vertex of P_2 dominates s_2 and the terminal vertex of P_m dominates s_m. Then

$$s_m s_1 C[y_1^+, y_2] P_m s_m \quad \text{and} \quad s_2 C[y_2^+, y_1] D_2 D_3 \dots D_{p-1} P_2 s_2$$

are vertex-disjoint cycles in D such that $s_m \in V(C_1)$.

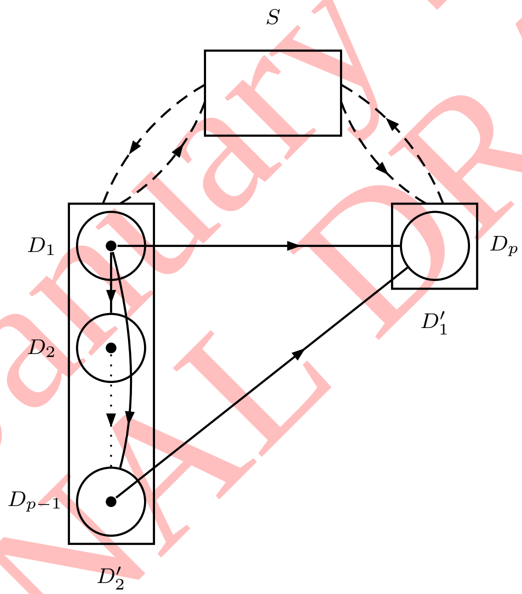


Fig. 3: The structure of D in Case 2.

Hence we may assume that $y_i = y_j$ for all $i, j \in \{1, 2, \dots, k\}$, which implies that there exists a vertex $y \in V(D_1)$ such that $S \to y$. Note that S is a transitive tournament (otherwise S contains a 3-cycle and we are done). Let $P = s_1 s_2 \dots s_k$ be the unique Hamiltonian path of S. Since $y_i = y_j$ for all i, j and $S \to y$, we have $(S - s_1) \to y^+$. It follows that D contains two vertex-disjoint paths from $\{y, y^+\}$ to $\{s_{k-1}, s_k\}$ and thus, we obtain two vertex-disjoint cycles C_1, C_2 in D by adding the appropriate arcs from $\{s_{k-1}, s_k\}$ to $\{y, y^+\}$. Note that each of the cycles C_1 and C_2 contains at least one vertex of D_1 and one vertex of S. Using Lemma 2.1, we can show that the remaining vertices in $D_p, D_{p-1}, \dots, D_2$ can be inserted in at least one of these cycles. It remains to show the same for the vertices of D_1 and S.

At first we consider the set S. Note that $s_i \to s_j$ for $i < j$ and that s_k and s_{k-1} have k and $k-1$ positive neighbors in D_1, respectively. In addition, recall that $N^-(s, D_p) \neq \emptyset$ for all $s \in S$. Using these observations and Lemma 2.1, all vertices of $S - \{s_{k-1}, s_k\}$ can be inserted in at least one of the cycles.

Now consider the set D_1. Assume that we have already inserted as much vertices as possible in C_1 and C_2. Let C be a Hamiltonian cycle of D_1 and let $C[v,w]$ be a path in D_1 such that $V(C[v,w]) \cap V(C_i) = \emptyset$ for $i = 1, 2$. Without loss of generality, w has a positive neighbor on C_1.

If $s_k \in V(C_1)$ (and $s_{k-1} \in V(C_2)$), we deduce that w and s are adjacent for every vertex $s \in S$, since $(S - s_k) \to s_k$. Because of the maximality assumption for $|V(C_1) \cup V(C_2)|$, we also know that $w \to (S \cap V(C_1))$. If there exists a vertex $s \in (S \cap V(C_2))$ that dominates w, the digraph D contains a 3-cycle and thus, is cycle complementary. It follows that $w \to S$ and hence $w \to (V(C_1) \cup V(C_2))$. But this implies that the path $C[v,w]$ can be inserted in C_2, a contradiction.

Otherwise $s_{k-1} \in V(C_1)$ (and $s_k \in V(C_2)$). Note that w particularly dominates $V(C_1) \cap V(D_1)$. Furthermore, the set $N^+(s_k, D_1) \cap V(C_1)$ is not empty and hence, w and s_k are adjacent. Now the same argumentation as above yields a contradiction.

Subcase 2.1.2: Suppose that $|V(D_p)| = 1$. The case that $p \geq 3$ can be solved analogously to Subcase 2.1.1. Therefore it remains to check the case that $p = 2$. Let C be a Hamiltonian cycle of D_1.

Subcase 2.1.2.1: Suppose there exist two indices $i \neq j$ such that $y_i \neq y_j$ and $s_i \to s_j$. Then we consider y_i^+ and y_j^+. If $y_i^+ = y_j$ or $y_j^+ = y_i$, the digraph D has a 3-cycle and it is immediate that D is cycle complementary. Otherwise note that s_i and y_j are adjacent. Therefore either $s_i \to y_j$ and $s_i y_j x_1^1 s_i$ is a 3-cycle in D or $y_j \to s_i$ and

$$C_1 = s_i C[y_i^+, y_j] s_i \quad \text{and} \quad C_2 = s_j C[y_j^+, y_i] x_1^1 s_j$$

are vertex-disjoint cycles in D such that $V(D) - (V(C_1) \cup V(C_2)) = S - \{s_i, s_j\}$. Now we can show analogously to Subcase 2.1.1 that D is cycle complementary.

Subcase 2.1.2.2: Suppose there exist two integers $i \neq j$ such that $y_i = y_j$ and $y_i^+ = y_j^+$ and neither s_i nor s_j can be inserted at another position of the Hamiltonian cycle C of D_1. Then, following Subcase 2.1.1 ($|V(D_p)| \geq 3$, $p \geq 3$ and $y_i = y_j$ for all $i, j \in \{1, 2, \ldots, k\}$), we see that D has complementary cycles.

Subcase 2.1.2.3: Suppose that $|\{y_1, y_2, \ldots, y_k\}| = k$ and $E(D[S]) = \emptyset$.

If $y_i^+ = y_j$ for some $i, j \in \{1, 2, \ldots, k\}$, the digraph D has a 3-cycle and we are done. Otherwise, since $k \geq 4$, there exist vertices $s_i \neq s_j$ in S such that x_1^1 has a negative neighbor $v_1 \neq y_i^+$ on $C[y_i^+, y_j^-]$ and a negative neighbor $v_2 \neq y_j^+$ on $C[y_j^+, y_i^-]$. Furthermore, we may assume, without loss of generality, that $y_i \to y_j$ and thus,

$$C_1 = s_i C[y_i^+, v_1] x_1^1 s_i \quad \text{and} \quad C_2 = s_j C[y_j^+, y_i] y_j s_j$$

are vertex-disjoint cycles in D. Consider the vertices on $C[v_1^+, y_1^-]$. Using Lemma 2.1, it follows that all these vertices can be inserted in C_2 unless there exists a vertex $u \in V(C[v_1^+, y_1^-])$ with the following properties: D contains a Hamiltonian cycle C_2' of $V(C_2) \cup V(C[u^+, y_1^-])$ and $u \to C_2'$. It follows that C_2' and $C_1' = s_i C[y_i^+, u] x_1^1 s_1$ are vertex-disjoint cycles in D that contain all vertices of D except $S - \{s_i, s_j\}$. Now let $m \notin \{i, j\}$. Since $x_1^1 \to s_m$ and $x_1^1 \in V(C_1)$, the vertex s_m can be inserted in C_1 if $N^+(s_m, C_1) \neq \emptyset$. Therefore we may assume that s_m has a positive neighbor on C_2' and thus, s_m can be inserted in C_2' unless $s_m \to C_2'$. But the latter implies that s_m and s_j are adjacent, a contradiction.

Subcase 2.2: Suppose that $k = 3$. First we show that the digraph D has a separating set $S = \{s_1, s_2, s_3\}$ such that $s_1 s_2 s_3 s_1$ is a 3-cycle in D. For this it suffices to show that D contains a 3-cycle. Following the proof of Subcase 2.1, in all cases except the last we either find a separating set S of D which has the appropriate condition or we see that D is cycle complementary. It remains to check Subcase 2.1.2.3 ($p = 2$, $|V(D_1)| \geq 3$, $|V(D_2)| = 1$, $|\{y_1, y_2, y_3\}| = 3$ and $E(D[S]) = \emptyset$). In addition, we may assume that $y_i^+ \neq y_j$ for all $i, j \in \{1, 2, 3\}$. Now we consider the vertices y_1, y_2

and y_3. Since $y_i \in N^-(x_1^1)$ for each $i \in \{1,2,3\}$, the subdigraph $D[\{y_1,y_2,y_3\}]$ is a tournament. We may assume, without loss of generality, that $y_1 \to y_2$. If y_3 is on $C[y_1^+, y_2]$, we can show that D is cycle complementary following the proof in Subcase 2.1.2.3. Therefore we may assume that y_3 is a vertex of the path $C[y_2^+, y_1]$. Analogously we deduce that D has the arcs y_3y_1 and y_2y_3 and thus, D contains the 3-cycle $y_1y_2y_3y_1$. Hence, we may assume that $S = \{s_1, s_2, s_3\}$ is a separating of D such that $s_1s_2s_3s_1$ is a 3-cycle.

Subcase 2.2.1: Suppose that $|V(D_p)| \geq 3$.

Subcase 2.2.1.1: Suppose that $|V(D_1)| \geq 3$ or $p \geq 3$.

Subcase 2.2.1.1.1: Assume that $|V(D_1)| \geq 3$ and there exists a vertex of S that dominates D_1, say $s_1 \to D_1$. Since $k = 3$, there exist vertices y_1, y_2, $y_3 \in V(D_1)$ such that $\{y_1, y_2, y_3\} \to D_2$ and at least one of these vertices, say y_1, dominates D_p. If S does not dominate D_1, we can choose y_1 such that $y_1 \to s_i \to y_1^+$, where $i = 2$ or $i = 3$. Furthermore, D has three non-incident arcs z_js_j, where $j = 1,2,3$, leading from D_p to S. Let C and C' be Hamiltonian cycles of D_1 and D_p, respectively. If $i = 2$, the cycles

$$s_3s_1C[y_2^+, y_1]C'[z_2^+, z_3]s_3 \quad \text{and} \quad s_2C[y_1^+, y_2]D_2D_3 \dots D_{p-1}C'[z_3^+, z_2]s_2$$

and if $i = 3$, the cycles

$$s_1C[y_2^+, y_1]C'[z_2^+, z_1]s_1 \quad \text{and} \quad s_2s_3C[y_1^+, y_2]D_2D_3 \dots D_{p-1}C'[z_1^+, z_2]s_2$$

are complementary in D.

Subcase 2.2.1.1.2: Assume that $|V(D_1)| \geq 3$ and no vertex of S dominates D_1. Then we deduce that $p \geq 3$ (otherwise D_2 and $D - V(D_2)$ are strong complementary subdigraphs of D). This case can be solved analogously to Subcase 2.1.1.

Subcase 2.2.1.1.3: Assume that $|V(D_1)| = 1$ and $p \geq 3$. Then $|V(D_2)| = 1$, since otherwise D_2 and $D - V(D_2)$ are strong complementary subdigraphs of D. We may assume, without loss of generality, that $s_1 \to D_2$ and thus,

$$s_1D_2D_3 \dots D_{p-1}C'[z_2^+, z_1]s_1 \quad \text{and} \quad s_2s_3D_1C'[z_1^+, z_2]s_2$$

are complementary cycles of D, where z_1, z_2 and C' are chosen as in Subcase 2.2.1.1.1.

Subcase 2.2.1.2: Suppose that $|V(D_1)| = 1$ and $p = 2$. Then $S \to D_1$ and $|V(D_2)| \geq 4$, since $|V(D)| \geq 8$. In addition, we have $|N^+(s_i, D_2)| \geq 1$ for each $i \in \{1,2,3\}$ and $|N^+(S, D_2)| \geq 2$. Furthermore, D has three non-incident arcs leading from D_2 to S. Let $C = b_1b_2 \dots b_tb_1$ be a Hamiltonian cycle of D_2, where $t \geq 4$. We may assume, without loss of generality, that D has the arcs b_1s_1, b_is_2 and b_js_3, where $2 \leq i \neq j \leq t$.

If there exists an arc b_qb_2 leading from $C[b_3, b_t]$ to b_2, the cycles

$$b_2b_3 \dots b_qb_2 \quad \text{and} \quad s_1s_2s_3x_1^2C[b_{q+1}, b_1]s_1$$

are complementary in D. Hence, since $k \geq 3$, at least one vertex of S dominates b_2. We consider the three cases $s_i \to b_2$ for $i \in \{1,2,3\}$.

Subcase 2.2.1.2.1: If $s_3 \to b_2$, the cycles

$$s_2s_3C[b_2, b_i]s_2 \quad \text{and} \quad s_1x_1^2C[b_{i+1}, b_1]s_1$$

are complementary in D.

Subcase 2.2.1.2.2: Suppose that $s_2 \to b_2$ and $s_3 \notin N^-(b_2)$. In this case

$$C_1 = s_2C[b_2, b_i]s_2 \quad \text{and} \quad C_2 = s_1x_1^2C[b_{i+1}, b_1]s_1$$

are vertex-disjoint cycles that contain all vertices of D but s_3.

If $b_m \to s_3 \to b_{m+1}$ for some index $m \notin \{1, i\}$, we can insert s_3 in one of these cycles and we are done.

Otherwise we deduce that $N^+(s_3, C[b_2, b_i]) = \emptyset$ and $b_i \to s_3 \to C[b_{i+1}, b_1]$. It follows that $2 \leq j \leq i-1$. We can analogously show that $N^+(s_1, C[b_{i+1}, b_j]) = \emptyset$ and $b_j \to s_1 \to C[b_{j+1}, b_i]$ and that $N^+(s_2, C[b_{j+1}, b_1]) = \emptyset$ and $b_1 \to s_2 \to C[b_2, b_j]$. Note that $s_1, s_3 \notin N^-(b_2)$. Hence $s_2 x_1^2 b_1 s_2$ is a 3-cycle and a separating of D such that the initial component of $D - \{s_2, x_1^2, b_1\}$ is the single vertex b_2. It follows that $b_2 \to \{s_1, s_3, b_3, b_4, \ldots, b_t\}$.

If $b_q s$ is an arc of D, where $3 \leq q \leq t-1$, the cycles

$$b_2 b_{q+1} b_{q+2} \ldots b_2 \quad \text{and} \quad s s^+ s^- x_1^2 C[b_3, b_q] s$$

are complementary in D.

Therefore we may assume that $N^-(S) = \{b_t, b_1, b_2\}$ (which implies that $j = 2$ and $i = t$). It follows that $C_3 = b_1 b_2 b_t b_1$ is a 3-cycle and a separating set of D. Furthermore, since $S \to x_1^2 \to C[b_3, b_{t-1}]$, the digraph $D - V(C_3)$ has at least three strong components. We have solved this case in Subcase 2.2.1.1.

Subcase 2.2.1.2.3: Suppose that $N^-(b_2) = \{b_1, s_1, x_1^2\}$. Then $N^-(b_2)$ induces a 3-cycle in D and is a separating set of D such that the initial component of $D - \{b_1, s_1, x_1^2\}$ is the single vertex b_2. Hence, we obtain complementary cycles of D following the argumentation in Subcase 2.2.1.2.2.

Subcase 2.2.2: Suppose that $|V(D_p)| = 1$. Since $|V(D)| \geq 8$ and $k = 3$, we have $|V(D_2^{'})| \geq 4$. Furthermore, $|N^+(D_{p-1}, S)| \geq 2$ and therefore we may assume, without loss of generality, that $\{s_1, s_2\} \subseteq N^+(D_{p-1})$.

Subcase 2.2.2.1: Suppose that $p \geq 3$ and $|V(D_1)| \geq 3$.

Subcase 2.2.2.1.1: Assume that at least two vertices of S dominate D_1, say $\{s_1, s_2\} \to D_1$. Then there exist two distinct vertices $y_1 \neq y_2$ in D_1 such that $y_1 \to x_1^1$ and $y_2 \to D_2$. If $s_3 \not\to D_1$, we can choose y_1 such that $y_1 \to s_3 \to y_1^+$. Furthermore, we may assume that $s_1 \in N^+(D_{p-1})$. Let C be a Hamiltonian cycle of D_1. It follows that

$$C_1 = s_1 C[y_1^+, y_2] D_2 D_3 \ldots D_{p-1} s_1 \quad \text{and} \quad C_2 = s_2 C[y_2^+, y_1] x_1^1 s_2$$

are vertex-disjoint cycles in D that include all vertices of D but s_3. Note that $s_3 \to s_1$. By Lemma 2.1, the vertex s_3 either can be inserted in C_1 or $s_3 \to C_1$. In the first case it is immediate that D is cycle complementary and in the latter case $s_2 \in N^+(D_{p-1})$. But then

$$s_2 C[y_1^+, y_2] D_2 D_3 \ldots D_{p-1} s_2 \quad \text{and} \quad s_3 s_1 C[y_2^+, y_1] x_1^1 s_3$$

show that D is cycle complementary.

Subcase 2.2.2.1.2: Assume that exactly one vertex of S, say s_1, dominates D_1.

If $s_2 \in N^+(D_{p-1})$, we choose a vertex $y_1 \in V(D_1)$ such that $y_1 \to s_2 \to y_1^+$. Let C be a Hamiltonian cycle of D_1. Then

$$s_3 s_1 C[y_2^+, y_1] x_1^1 s_3 \quad \text{and} \quad s_2 C[y_1^+, y_2] D_2 D_3 \ldots D_{p-1} s_2$$

are complementary cycles of D.

Otherwise we have $N^+(D_{p-1}) = \{x_1^1, s_1, s_3\}$. Now we choose y_1 such that $y_1 \to s_3 \to y_1^+$ and we consider

$$C_1 = s_1 C[y_2^+, y_1] x_1^1 s_1 \quad \text{and} \quad C_2 = s_3 C[y_1^+, y_2] D_2 D_3 \ldots D_{p-1} s_3.$$

These cycles are vertex-disjoint and contain all vertices of D except s_2. It follows that $s_2 \to D_j$ for $j = 2, 3, \ldots, p-1$ (otherwise C_2 can be extended by s_2). Hence

$$s_2 D_2 D_3 \ldots D_{p-1} x_1^1 s_2 \quad \text{and} \quad s_3 s_1 C[y_1^+, y_1] s_3$$

are complementary cycles of D.

Subcase 2.2.2.1.3: Assume that all vertices of S can be inserted in the Hamiltonian cycle D_1. This case can be solved analogously to Subcase 2.1.2.

Subcase 2.2.2.2: Suppose that $p \geq 3$ and $|V(D_1)| = 1$. Then $S \to x_1^2$ and $|V(D_2)| = 1$, since otherwise D_2 and $D - V(D_2)$ are strong complementary subdigraphs of D. Since $k \geq 3$, at least one vertex of S, say s_1, dominates x_2^2.

If $s_1 \in N^+(D_{p-1})$, the cycles

$$s_1 D_2 D_3 \ldots D_{p-1} s_1 \quad \text{and} \quad s_2 s_3 x_1^2 x_1^1 s_2$$

are complementary in D.

Otherwise we have $N^+(D_{p-1}) = \{x_1^1, s_2, s_3\}$ and

$$s_3 s_1 D_2 D_3 \ldots D_{p-1} s_3 \quad \text{and} \quad s_2 x_1^2 x_1^1 s_2$$

show that D is cycle complementary.

Subcase 2.2.2.3: Suppose that $p = 2$. Then $|V(D_1)| \geq 4$. Let $C = a_1 a_2 \ldots a_q a_1$ be a Hamiltonian cycle of D_1, where $q \geq 4$. Since $k = 3$, $|V(D_2)| = 1$ and $s_1 s_2 s_3 s_1$ is a 3-cycle in D, all vertices of S can be inserted in C.

Subcase 2.2.2.3.1: Suppose that D_1 contains vertices a_i, a_j, a_m such that $a_i \to s_1 \to a_{i+1}$, $a_j \to s_2 \to a_{j+1}$ and $a_m \to s_3 \to a_{m+1}$ and $1 \leq i < j < m \leq q$. Then, since $q \geq 4$, we may assume, without loss of generality, that $i + 1 \neq j$. Note that s_1 and a_j are adjacent.

If D has the arc $a_j s_1$, the cycles

$$s_1 C[a_{i+1}, a_j] s_1 \quad \text{and} \quad s_2 C[a_{j+1}, a_m] s_3 C[a_{m+1}, a_i] x_1^1 s_2$$

are complementary in D.

Otherwise $s_1 \to a_j$ and $C_3 = s_1 a_j x_1^1 s_1$ is a 3-cycle in D. If $D - V(C_3)$ is strong, we are done. If $|N^+(x, D - V(C_3))| \geq 1$ for all vertices $x \in V(D) - V(C_3)$, the terminal component of $D - V(C_3)$ is not a single vertex. We have solved this case in Subcase 2.2.1. Hence, we may assume that $N^+(a_{j-1}) = V(C_3)$. If $a_i \to a_j$, the cycles

$$s_1 C[a_{i+1}, a_{j-1}] s_1 \quad \text{and} \quad s_2 C[a_{j+1}, a_m] s_3 C[a_{m+1}, a_i] a_j x_1^1 s_2$$

are complementary in D. Therefore we may assume that D has the arc $a_j a_i$. Now we consider the 3-cycle $C_3^{'} = s_1 a_j a_i s_1$. Following the argumentation above, we deduce that $N^+(a_{i-1}) = V(C_3^{'})$. But then

$$a_i a_{i+1} \ldots a_j a_i \quad \text{and} \quad s_1 s_2 C[a_{j+1}, a_m] s_3 C[a_{m+1}, a_{i-1}] x_1^1 s_1$$

show that D is cycle complementary. We can analogously solve the case $1 \leq i < m < j \leq q$.

Subcase 2.2.2.3.2: Suppose that a_i, a_j and a_m can be chosen such that $|\,\{i, j, m\}\,| = 2$, but not such that $|\,\{i, j, m\}\,| = 3$. We may assume, without loss of generality, that $a_i = a_j$ and $a_{i+1} = a_{j+1}$. It follows that s_3 and a_i are adjacent.

If $a_i \to s_3$, the cycles

$$s_3 C[a_{m+1}, a_i] s_3 \quad \text{and} \quad s_1 s_2 C[a_{i+1}, a_m] x_1^1 s_1$$

are complementary in D.

Othwerwise $s_3 \to a_i$ and $C_3 = s_3 a_i x_1^1 s_3$ is a 3-cycle in D. Like in Subcase 2.2.2.3.1 it follows that $N^+(a_{i-1}) = V(C_3)$ and thus,

$$s_2 s_3 a_i s_2 \quad \text{and} \quad s_1 C[a_{i+1}, a_{i-1}] x_1^1$$

show that D is cycle complementary.

Subcase 2.2.2.3.3: Suppose that a_i, a_j and a_m can be chosen such that $|\{i, j, m\}| = 1$, but not such that $|\{i, j, m\}| > 1$. This case can be solved analogously to Subcase 2.1.2.

Subcase 2.3: Suppose that $k = 2$.

Subcase 2.3.1: Suppose that $|V(D_p)| \geq 3$. Note that the case $p \geq 3$ and $|V(D_1)| \geq 3$ can be solved analogously to Subcase 2.2.

Subcase 2.3.1.1: Suppose that $p \geq 3$ and $|V(D_1)| = 1$. Then $D[S]$ is a tournament and we may assume, without loss of generality, that $s_1 \to s_2$.

If $|V(D_2)| \geq 3$, it is easy to see that D_2 and $D - V(D_2)$ are complementary strong subdigraphs of D.

Otherwise we have $|V(D_2)| = 1$. Since $|N^-(D_2)| \geq 2$ and $N^-(D_2) \subseteq (V(D_1) \cup S)$, it is immediate that D_2 has at least one negative neighbor s_i in S. Let P_1 and P_2 be complementary paths of D_p such that the last vertex of P_j dominates s_j for $j = 1, 2$. Then

$$s_{3-i}x_1^2P_{3-i}s_{3-i} \quad \text{and} \quad s_iD_2D_3 \dots D_{p-1}P_is_i$$

are complementary cycles of D.

Subcase 2.3.1.2: Suppose that $p = 2$ and $|V(D_1)| \geq 3$.

If both s_1 and s_2 have positive and negative neighbors in D_1, a Hamiltonian cycle of D_1 can be extended by s_1 and s_2. This extended cycle and a Hamiltonian cycle of D_2 are complementary cycles of D.

Therefore we may assume that at least one vertex of S dominates D_1. If $S \to D_1$, the digraph D is cycle complementary, since $|N^-(D_2, D_1)|, |N^-(S, D_2)| \geq 2$. Otherwise we assume that $s_i \to D_1$ and that s_{3-i} has positive and negative neighbors in D_1 for an index $i \in \{1, 2\}$. It follows that D_1 contains vertices $y_1 \neq y_2$ such that $y_1 \to D_2$ and $y_2 \to s_{3-i} \to y_2^+$. Let P_1 and P_2 be complementary paths of D_p such that the last vertex of P_j dominates s_j for $j = 1, 2$ and let C be a Hamiltonian cycle of D_1. Then

$$s_iC[y_1^+, y_2]P_is_i \quad \text{and} \quad s_{3-i}C[y_2^+, y_1]P_{3-i}s_{3-i}$$

are complementary cycles of D.

Subcase 2.3.1.3: Suppose that $p = 2$ and $|V(D_1)| = 1$. Then $|V(D_2)| \geq 5$, since $|V(D)| \geq 8$. Furthermore, $D[S]$ is a tournament and hence we may assume, without loss of generality, that $s_1 \to s_2$. Let $C = b_1b_2 \dots b_tb_1$ be a Hamiltonian cycle of D_2, where $t \geq 5$. Since $k = 2$, we have $|N^-(s_1, D_2)| \geq 2$, $|N^+(s_2, D_2)| \geq 1$ and $|N^-(s_2, D_2)| \geq 1$. Therefore we may assume, without loss of generality, that D has the arcs b_1s_1, b_is_2 and s_2b_{i+1}, where $i \neq 1$. It follows that D has no arc b_qb_2 leading from $C[b_3, b_t]$ to b_2, because otherwise

$$b_qb_2b_3 \dots b_q \quad \text{and} \quad s_1s_2x_1^2C[b_{q+1}, b_1]s_1$$

are complementary cycles of D.

Subcase 2.3.1.3.1: Suppose that $b_{i+1} \notin N^-(s_1)$. We may assume, without loss of generality, that s_1 has no negative neighbor on $C[b_{i+1}, b_t]$. Considering $D - b_1$, it is immediate that D has an arc leading from $C[b_{i+1}, b_t]$ to $\{b_2, b_3, \dots, b_i, s_1, s_2, x_1^2\}$, since D is 2-connected.

If D has an arc b_js_2, where $i + 2 \leq j \leq t$, we obtain $s_1 \to \{b_{i+1}, b_{i+2}, \dots, b_j\}$ because of the choice of b_1. It follows that

$$s_1C[b_{i+1}, b_1]s_1 \quad \text{and} \quad s_2x_1^2C[b_2, b_i]s_2$$

are complementary cycles of D.

Otherwise D has an arc b_jb_m, where $i + 1 \leq j \leq t$ and $3 \leq m \leq i$. In this case

$$C_1 = s_1x_1^2C[b_{j+1}, b_1]s_1 \quad \text{and} \quad C_2 = s_2C[b_{i+1}, b_j]C[b_m, b_i]s_2$$

are vertex-disjoint cycles in D. Using Lemma 2.1, it follows that there exists a vertex b_q, where $2 \leq q \leq m-1$, such that all vertices of $C[b_{q+1}, b_{m-1}]$ can be inserted in C_2 and $b_q \rightarrow V(C_2) \cup V(C[b_{q+1}, b_{m-1}])$. Hence, D has particularly the arc $b_q b_{i+1}$ and thus,

$$C_1^{'} = b_q C[b_{i+1}, b_q] \quad \text{and} \quad C_2^{'} = s_2 x_1^2 C[b_{q+1}, b_i] s_2$$

are vertex-disjoint cycles in D such that $C_1^{'}$ and $C_2^{'}$ contain all vertices of D except s_1. Since $s_1 \rightarrow x_1^2$, we conclude that $s_1 \rightarrow C_2^{'}$ which implies that D has the arc $s_1 b_m$. Because of the choice of b_1, it now follows that $s_1 \rightarrow C[b_{i+1}, b_j]$ and thus, the cycles

$$s_1 C[b_{i+1}, b_1] s_1 \quad \text{and} \quad s_2 x_1^2 C[b_2, b_i] s_2$$

show that D is cycle complementary.

Subcase 2.3.1.3.2: Suppose that $b_{i+1} \in N^-(s_1)$. Then we may assume, without loss of generality, that $b_{i+1} = b_1$. Since $|N^-(s_1, D_2)| \geq 2$, the vertex s_1 has a negative neighbor $b_j \neq b_1$. Note that s_1 and b_t are adjacent.

Subcase 2.3.1.3.2.1: Assume that $b_t \rightarrow s_1$. Then we consider $D - \{s_1, b_t\}$.

If D has an arc bs_2 such that $b \notin \{b_t, b_1, s_1, x_1^2\}$, the cycles

$$s_2 C[b_1, b] s_2 \quad \text{and} \quad s_1 x_1^2 C[b^+, b_t] s_1$$

are complementary in D.

Otherwise $\{s_1, b_t\}$ is a separating set of D. Since $s_2 \rightarrow x_1^2 \rightarrow (D - \{b_t, s_1, s_2\})$, the digraph $D - \{s_1, b_t\}$ has at least three strong components and the first strong component has only one vertex. We already have solved this case in Subcase 2.3.1.1.

Subcase 2.3.1.3.2.2: Assume that $s_1 \rightarrow b_t$. Following the argumentation in Subcase 2.3.1.3.2.1, we deduce that D has an arc bs_2 such that $b \notin \{b_t, b_1, s_1, x_1^2\}$. Now we consider $D^{'} := D - \{b_1, x_1^2\}$. In the following we will show that $N^-(b_2) \neq \{b_1, x_1^2\}$. Assume to the contrary that $N^-(b_2) = \{b_1, x_1^2\}$ which implies that the initial component of $D^{'}$ is the single vertex b_2. It follows that $b_2 \rightarrow \{b_3, b_4, \ldots, b_t, s_1, s_2\}$.

If s_2 has a negative neighbor $b \notin \{b_2, b_t\}$, the cycles

$$C_1 = b_2 C[b^+, b_2] \quad \text{and} \quad C_2 = s_2 x_1^2 C[b_3, b] s_2$$

are vertex-disjoint and contain all vertices of D except s_1. If s_1 can be inserted in C_2, we are done. Otherwise $s_1 \rightarrow C_2$ and thus,

$$s_1 C[b_3, b_1] s_1 \quad \text{and} \quad s_2 x_1^2 b_2 s_2$$

are complementary cycles of D.

Otherwise we have $N^-(s_2, D_2) = \{b_2, b_t\}$. In this case we consider $D^{''} := D - \{b_1, b_2\}$. If $N^-(s_1) = \{b_1, b_2\}$, the initial component of $D^{''}$ is the single vertex s_1. It follows that $s_1 \rightarrow \{b_3, b_4, \ldots, b_t, s_2, x_1^2\}$ and thus,

$$s_2 x_1^2 b_2 s_2 \quad \text{and} \quad s_1 C[b_3, b_1] s_1$$

are complementary cycles of D. Therefore we assume that there exists an index $3 \leq m \leq t-1$ such that $b_m \rightarrow s_1 \rightarrow C[b_{m+1}, b_t]$. But then

$$b_2 C[b_{m+1}, b_t] s_2 b_1 b_2 \quad \text{and} \quad s_1 x_1^2 C[b_3, b_m] s_1$$

show that D is cycle complementary.

All in all we have shown that $N^-(b_2) \neq \{b_1, x_1^2\}$. If D has the arc s_2b_2, the cycles

$$s_2C[b_2, b_t]s_2 \quad \text{and} \quad s_1x_1^2b_1s_1$$

are complementary in D. It remains to check the case that D has the arc s_1b_2. Since $|N^-(s_1, D_2)| \geq 2$, there exists an integer $3 \leq m \leq t-1$ such that $b_m \to s_1 \to C[b_{m+1}, b_t]$. In addition, following the argumentation in Subcase 2.3.1.3.2.1, D has an arc bs_2 such that $b \notin \{b_t, b_1, s_1, x_1^2\}$. The vertex-disjoint cycles

$$C_1 = s_1C[b_2, b_m]s_1 \quad \text{and} \quad C_2 = s_2x_1^2C[b_{m+1}, b_t]s_2$$

contain all vertices of D but b_1. It follows that $b_1 \to C_1$, since b_1 can be inserted in C_1 otherwise. But now

$$s_2x_1^2C[b_2, b]s_2$$

and

$$\begin{cases} s_1C[b_{m+1}, b_1]C[b^+, b_m]s_1 & \text{if } b \in \{b_2, b_3, \ldots, b_m\} \\ s_1C[b^+, b_1]s_1 & \text{if } b \in \{b_{m+1}, b_{m+2}, \ldots, b_{t-1}\} \end{cases}$$

are complementary cycles of D.

Subcase 2.3.2: Suppose that $|V(D_p)| = 1$.

Subcase 2.3.2.1: Suppose that $p \geq 3$ and $|V(D_1)| \geq 3$. Then $|N^+(D_{p-1}, S)| \geq 1$ and, in addition, D has two non-incident arcs leading from $V(D_{p-1}) \cup \{x_1^1\}$ to S.

Subcase 2.3.2.1.1: Assume that $S \to D_1$. Then $D[S]$ is a tournament and hence we can assume, without loss of generality, that D has the arc s_1s_2. Furthermore, there exist vertices $y_1 \neq y_2$ in D_1 such that $y_1 \to D_2$ and $y_2 \to x_1^1$. Now it is easy to see that D is cycle complementary.

Subcase 2.3.2.1.2: Assume that $s_1 \to D_1$ and $N^-(s_2, D_1) \neq \emptyset$. Then D_1 contains vertices $y_1 \neq y_2$ such that $y_1 \to D_2$ and $y_2 \to x_1^1$. In addition, y_2 can be chosen such that $y_2 \to s_2 \to y_2^+$.

If $s_2 \in N^+(D_{p-1})$, let C be a Hamiltonian cycle of D_1. Then

$$s_1C[y_1^+, y_2]x_1^1s_1 \quad \text{and} \quad s_2C[y_2^+, y_1]D_2D_3 \ldots D_{p-1}s_2$$

are complementary cycles of D.

Otherwise we deduce that $N^+(D_{p-1}) = \{s_1, x_1^1\}$. If $|V(D_{p-1})| \geq 3$, the digraph D has two non-incident arcs z_1s_1, $z_2x_1^1$ leading from D_{p-1} to $\{s_1, x_1^1\}$. Let C' be a Hamiltonian cycle of D_{p-1}. Then

$$s_1C[y_1^+, y_2]C^{'}[z_2^+, z_1]s_1 \quad \text{and} \quad s_2C[y_2^+, y_1]D_2D_3 \ldots D_{p-2}C^{'}[z_1^+, z_2]x_1^1s_2$$

are complementary cycles of D. Therefore we may assume that $V(D_{p-1}) = \{z\}$. Note that $z \to s_1$ or $z \to s_2$.

Subcase 2.3.2.1.2.1: Suppose that $p \geq 4$. It follows that $x_1^1 \in N^+(D_{p-2})$. Hence

$$s_1C[y_1^+, y_2]zs_1 \quad \text{and} \quad s_2C[y_2^+, y_1]D_2D_3 \ldots D_{p-2}x_1^1s_2$$

are complementary cycles of D.

Subcase 2.3.2.1.2.2: Suppose that $p = 3$ and $z \to s_2$. Then

$$s_2C[y_2^+, y_1]zs_2 \quad \text{and} \quad s_1C[y_1^+, y_2]x_1^1s_1$$

are complementary cycles of D.

Subcase 2.3.2.1.2.3: Suppose that $p = 3$ and $z \to s_1$.

If there exists a vertex $y \neq y_2$ in D_1 such that $y \to x_1^1$, the cycles

$$s_2C[y_2^+, y]x_1^1s_2 \quad \text{and} \quad s_1C[y^+, y_2]zs_1$$

show that D is cycle complementary.

Otherwise $\{y_2, z\}$ is a separating set of D such that the initial strong component of $D - \{y_2, z\}$ is the single vertex x_1^1. Since there is no arc between x_1^1 and $D_1 - y_2$, the decomposition of $D - \{y_2, z\}$ according to Theorem 2.6 has at least three strong components. This case was solved in Subcase 1.

Subcase 2.3.2.1.3: Assume that $N^-(s_i, D_1) \neq \emptyset$ for each $i \in \{1, 2\}$. We may assume, without loss of generality, that $s_2 \in N^+(D_{p-1})$. Let C be a Hamiltonian cycle of D_1.

If there exist vertices $y_1 \neq y_2$ in D_1 such that $y_i \to s_i \to y_i^+$ for $i \in \{1, 2\}$, the cycles

$$s_1 C[y_1^+, y_2] x_1^1 s_1 \quad \text{and} \quad s_2 C[y_2^+, y_1] D_2 D_3 \ldots D_{p-1} s_2$$

are complementary in D.

The case that there is no pair $y_1 \neq y_2$ of vertices in D_1 such that $y_i \to s_i \to y_i^+$ for $i \in \{1, 2\}$ can be solved analogously to Subcase 2.1.1.

Subcase 2.3.2.2: Suppose that $p \geq 4$ and $|V(D_1)| = 1$. Then we conclude that $N^-(D_2, S) \neq \emptyset$ and $N^+(D_{p-1}, S) \neq \emptyset$. Therefore we may assume that $s_i \in N^-(D_2)$ for an index $i \in \{1, 2\}$. Note that $|V(D_2)| = 1$, since otherwise D_2 and $D - V(D_2)$ are strong complementary subdigraphs of D.

If $s_i \in N^+(D_{p-1})$, the cycles

$$s_i D_2 D_3 \ldots D_{p-1} s_i \quad \text{and} \quad s_{3-i} x_1^2 x_1^1 s_{3-i}$$

show that D is cycle complementary.

The remaining case that $N^+(D_{p-1}) = \{s_{3-i}, x_1^1\}$ can be solved analogously to Subcase 2.3.2.1.

Subcase 2.3.2.3: Suppose that $p = 3$ and $|V(D_1)| = 1$. Then $|V(D_2)| \geq 4$, since $|V(D)| \geq 8$. Furthermore, we may assume, without loss of generality, that $s_1 \to s_2$. But now a Hamiltonian cycle of D_2 and $s_1 s_2 x_1^2 x_1^1 s_1$ are complementary cycles of D.

Subcase 2.3.2.4: Suppose that $p = 2$. Then $|V(D_1)| \geq 5$, since $|V(D)| \geq 8$. In addition, at most one vertex of S dominates D_1. Note that every vertex of D has at least three negative neighbors, since otherwise D is cycle complementary by the case $|V(D_1)| = 1$.

Subcase 2.3.2.4.1: Suppose that there exists a vertex $s \in S$ such that $s \to D_1$. Then we may assume, without loss of generality, that $s_1 \to D_1$ and $N^-(s_2, D_1) \neq \emptyset$. It follows that D has the arc $s_2 s_1$. Furthermore, D_1 contains vertices $y_1 \neq y_2$ such that $y_1 \to x_1^1$ and $y_2 \to s_2 \to y_2^+$. Since $|N^-(s_2)| \geq 3$, there is a vertex $y \neq y_2$ in D_1 such that $y \to \{s_2, x_1^1\}$. Let C be a Hamiltonian cycle of D_1. Then

$$s_2 C[y_2^+, y] s_2 \quad \text{and} \quad s_1 C[y^+, y_2] x_1^1 s_1$$

are complementary cycles of D.

Suppose now that neither s_1 nor s_2 dominates D_1. Note that we can solve the case that y_1 cannot be chosen unequal to y_2 analogously to Subcase 2.1.1. We consider the following cases.

Subcase 2.3.2.4.2: Suppose that y_1 and y_2 can be chosen such that, without loss of generality, $y_1^+ = y_2$, but not such that $y_i^+ \neq y_{3-i}$ for each $i \in \{1, 2\}$. Let C be a Hamiltonian cycle of D_1.

Subcase 2.3.2.4.2.1: Assume that $s_2 \to s_1$. Then D has the arc $s_2 y_1$, since otherwise

$$s_1 y_2 x_1^1 s_1 \quad \text{and} \quad s_2 C[y_2^+, y_1] s_2$$

are complementary cycles of D. It follows that $s_2 \to D_1 - y_2$, a contradiction to the fact that $|N^-(s_2)| \geq 3$.

Subcase 2.3.2.4.2.2: Assume that $s_1 \to s_2$. We consider the positive neighborhood of s_1 and the negative neighborhood of s_2.

If $T := N^+(s_1) = \{s_2, y_2\}$, the digraph $D - T$ has at least three strong components s_1, x_1^1 and $D_1 - y_2$ and thus, D is cycle complementary by one of the Subcases 2.3.2.1, 2.3.2.2 or 2.3.2.3.

If $U := N^-(s_2) = \{s_1, x_1^1, y_2\}$, the set U is a minimal separating set of D. It follows that $\{s_2\}$ is the initial component of $D - U$ and thus, $s_2 \rightarrow D_1 - U$. Let $y \neq y_1$ be a negative neighbor of s_1 in D_1. Then

$$s_1 C[y_1^+, z_1] s_1 \quad \text{and} \quad s_2 C[z_1^+, y] x_1^1 s_1$$

are complementary cycles of D.

All in all we may assume that $|N^+(s_1)| \geq 3$ and $|N^-(s_2)| \geq 4$. It follows that D has the arc $s_1 y_2^+$. In addition, s_2 dominates the successor of y_2^+ on C and has a negative neighbor $z_2 \notin \{x_1^1, s_1, y_2\}$. Note that y_2^+ has a positive neighbor w besides its successor on C.

If $w = x_1^1$, the cycles

$$s_1 y_2^+ x_1^1 s_1 \quad \text{and} \quad s_2 C[y_2^{++}, y_2] s_2$$

are complementary in D.

If w is on $C[z_2^+, y_1]$, the cycles

$$C_1 = s_1 y_1^+ y_2^+ C[w, y_1] s_1 \quad \text{and} \quad C_2 = s_2 C[y_2^{++}, z_2] x_1^1 s_2$$

are vertex-disjoint. Using Lemma 2.1, it follows that if D is not cycle complementary, there is a vertex $u \in V(C[z_2^+, w^-])$ such that $D[\{s_1, x_1^1\} \cup V(C[u^+, y_2^+])]$ has a Hamiltonian cycle C' and $u \rightarrow C'$. But then C' and $s_2 C[y_2^{++}, u] x_1^1 s_2$ show that D is cycle complementary.

If w is on $C[y_2^{++}, z_2]$, either $y_2^+ \rightarrow C[y_2^+, w]$ or there exists a vertex $u \in V(C[y_2^+, w])$ such that $u \rightarrow y_2^+ \rightarrow u^+$. The latter implies that s_i and u are adjacent for each $i = 1, 2$ and thus, $s_i \rightarrow C[y_2^+, u]$ for $i = 1, 2$. Now we can apply the same arguments as above on u^- and $N^+(u^-)$ instead of y_2^+ and $N^+(y_2^+)$. In the former case we can apply the same arguments as above on w^- and $N^+(w^-)$ instead of y_2^+ and $N^+(y_2^+)$. By doing this, we obtain complementary cycles of D in a finite number of steps.

Subcase 2.3.2.4.2.3: Assume that s_1 and s_2 are not adjacent. Recall that s_1 has at least one positive neighbor in D_1 besides y_1^+ and that neither s_1 nor s_2 can be inserted at another position in C. It is easy to see that these observations lead to a contradiction to the fact that s_1 and s_2 are not adjacent.

Subcase 2.3.2.4.3: Suppose that y_1 and y_2 can be chosen such that $y_1 \neq y_2$ and $y_i^+ \neq y_{3-i}$ for each index $i = 1, 2$. In this case we may assume, without loss of generality, that D has the arc $y_2 y_1$. Note that if x_1^1 has a negative neighbor on the path $C[y_2^+, y_1^-]$, with the help of Lemma 2.1 it is easy to check that D is cycle complementary (choose $y \in N^-(x_1^1)$ such that $C[y, y_1^-]$ has minimal length).

Subcase 2.3.2.4.3.1: Assume that s_1 and s_2 are not adjacent.

Assume that there is an arc $u y_1^+$ in D such that u is on $C[y_2^+, y_1^-]$. Then u and s_1 are adjacent. Due to the observations above, it follows that $s_1 \rightarrow C[u, y_2^+]$. Hence, s_1 and s_2 are adjacent, a contradiction.

Considering $D - y_1$, it is easy to see that D has an arc leading from $\{s_2\} \cup V(C[y_2^+, y_1^-])$ to $C[y_1^+, y_2]$. Let $v_1 v_2$ be such an arc such that $C[y_1^+, v_2]$ has minimal length and, under this condition, $C[v_1, y_1]$ has minimal length. Let $v_2^- = v_3$. Since $N^-(S, C[y_2^+, y_1^-]) = \emptyset$ and $E(D[S]) = \emptyset$, we conclude that $v_3 \rightarrow C[y_2^+, v_1]$ and $v_3 \rightarrow s_2$.

Now we consider the vertex-disjoint cycles

$$C_1 = s_1 C[y_1^+, v_3] x_1^1 s_1 \quad \text{and} \quad C_2 = s_2 C[y_2^+, v_1] C[v_2, y_2] s_2.$$

If $v_1 = y_1$ or if all vertices of the path $C[v_1^+, y_1]$ can be inserted in C_1, it is immediate that D is cycle complementary. Otherwise there exists a vertex on $C[v_1^+, y_1]$ that dominates C_1. Since S

has no negative neighbors on $C[v_1^+, y_1^-]$, it follows that this vertex is y_1, i.e., $y_1 \to C_1$. Because $|N^-(v_2)| \geq 3$, there exists a negative neighbor w of v_2 such that $w \notin \{v_1, v_3\}$. Note that w is not on $C[v_1^+, y_1]$.

It easy to check that D is cycle complementary if $w \in \{s_1, s_2\} \cup V(C[y_2, v_1^-]) \cup V(C[y_1^+, v_3^-])$. Hence w is on $C[v_2^+, y_2^-]$ and thus, D contains the vertex-disjoint cycles

$$C_1' = s_1 C[y_1^+, v_3] s_2 C[y_2^+, y_1] s_1 \quad \text{and} \quad C_2' = C[v_2, w] v_2.$$

Note that the vertex y_2 can be inserted in $C_1^{'}$. Using Lemma 2.1, it follows that there exists a vertex $u \in V(C[w^+, y_2^-])$ such that all vertices of the path $C[u^+, y_2]$ can be inserted in $C_1^{'}$ and $u \to A := V(C_1^{'}) \cup V(C[u^+, y_2])$. Therefore u^+ has a negative neighbor $z \in A$. Now it is easy to check the cycle complementarity of D.

Subcase 2.3.2.4.3.2: Assume that $s_2 \to s_1$. Since S has no negative neighbor on $C[y_2^+, y_1]$, it follows that $s_2 \to C[y_2^+, y_1]$.

Considering $D - y_1$, it is easy to see that D has an arc $v_1 v_2$ leading from $C[y_2^+, y_1^-]$ to $C[y_1^+, y_2]$. Now we consider the vertex-disjoint cycles

$$C_1 = C[v_2, v_1] v_2 \quad \text{and} \quad C_2 = s_2 C[v_1^+, y_1] x_1^1 s_2.$$

Using Lemma 2.1, it follows that there exists a vertex $v_3 \in V(C[y_1^+, v_2^-]) \cup \{s_1\}$ that dominates $V(C_1) \cup V(C[v_3^+, v_2^-])$.

If $v_3 = s_1$, let w be a negative neighbor of s_2 on $C[y_1^+, y_2^-]$. It follows that

$$s_1 C[w^+, y_2] x_1^1 s_1 \quad \text{and} \quad s_2 C[y_2^+, w] s_2$$

are complementary cycles of D.

If $v_3 \neq s_1$, the vertices s_2 and v_3 are adjacent. If D has the arc $v_3 s_2$, the cycle

$$C^{'} = s_2 C[v_1^+, y_1] s_1 C[y_1^+, v_3] x_1^1 s_2$$

and a Hamiltonian cycle of $D[V(C_1) \cup V(C[v_3^+, v_2^-])]$ are complementary cycles of D. Therefore we assume now that D has the arc $s_2 v_3$. Recall that s_2 has at least three negative neighbors and thus, a negative neighbor $z_2 \neq y_2$. We now consider the possibilities $z_2 \in V(C[v_2, y_2^-])$, $z_2 \in V(C[v_3^+, v_2^-])$ and $z_2 \in V(C[y_1^+, v_3^-])$. In the first two cases we choose $z_2 \in N^-(s_2)$ such that $C[z_2, y_2]$ has maximal length.

Subcase 2.3.2.4.3.2.1: Suppose that $z_2 \in V(C[v_2, y_2^-])$. In this case we consider the vertex-disjoint cycles

$$C_1 = s_1 C[y_1^+, v_3] C[z_2^+, y_2] y_1 x_1^1 s_1 \quad \text{and} \quad C_2 = s_2 C[y_2^+, v_1] C[v_2, z_2] s_2.$$

Since $N^-(S, C[y_2^+, y_1^-]) = \emptyset$, the vertices of the path $C[v_1^+, y_1^-]$ can be inserted in C_1. If the vertices of $C[v_3^+, z_2^-]$ cannot be inserted in C_2, there exists a vertex u on $C[v_3^+, z_2^-]$ such that $u \to C[u^+, v_1]$ and $u \to s_2$ by Lemma 2.1. In addition, $V(C_2) \cup V(C[u^+, v_2^-])$ induces a Hamiltonian subdigraph of D. But then

$$C_1' = s_1 C[y_1^+, u] x_1^1 s_1 \quad \text{and} \quad C_2' = s_2 C[y_2^+, v_1] C[v_2, y_2] s_2$$

are vertex-disjoint cycles of D such that the vertices of $C[v_1^+, y_1]$ can be inserted in C_1' and the vertices of $C[u^+, v_2^-]$ can be inserted in C_2'. It follows that D is cycle complementary.

Subcase 2.3.2.4.3.2.2: Suppose that $z_2 \in V(C[v_3^+, v_2^-])$. We consider the vertex-disjoint cycles

$$C_1 = s_1 C[y_1^+, z_2] x_1^1 s_2 C[v_1^+, y_1] s_1 \text{ and } C_2 = v_1 C[v_2, v_1].$$

Using Lemma 2.1, the vertices on $C[z_2^+, v_2^-]$ can be inserted in C_2.

Subcase 2.3.2.4.3.2.3: Suppose that $z_2 \in V(C[y_1^+, v_3^-])$. In this case we choose z_2 such that $z_2 \to s_2 \to C[z_2^+, v_3]$. Then we consider the vertex-disjoint cycles

$$C_1 = s_1 C[y_1^+, z_2] x_1^1 s_1 \quad \text{and} \quad C_2 = s_2 C[y_2^+, v_1] C[v_2, y_2] s_2.$$

Note that all vertices of the path $C[z_2^+, v_2^-]$ can be inserted in C_2 by using Lemma 2.1.

Since s_1 has no negative neighbor on $C[v_1^+, y_1]$, it follows that $y_1 \to C[y_1^+, z_2]$. Because $|N^-(z_2^+)| \geq 3$, the vertex z_2^+ has a negative neighbor $v_4 \notin \{s_2, z_2\}$. It is easy to check that D is cycle complementary if $v_4 \in \{s_1\} \cup V(C[y_2, z_2])$. Therefore we may assume that $v_4 \in V(C[z_2^+, y_2^-])$. But then

$$C_1' = s_2C[v_1^+, y_1]s_1C[y_1^+, z_2]s_2 \quad \text{and} \quad C_2' = v_4C[z_2^+, v_3]C[v_4^+, v_1]C[v_2, v_4]$$

are vertex-disjoint cycles in D such that the remaining vertices on $C[v_3^+, v_2^-]$ can be inserted in C_2. Hence, D is cycle complementary.

Subcase 2.3.2.4.3.3: Assume that $s_1 \to s_2$. It follows that s_1 and y_2 are adjacent.

If $y_2 \to s_1$, the cycles

$$s_1C[y_1^+, y_2]s_1 \quad \text{and} \quad s_2C[y_2^+, y_1]x_1^1s_2$$

are complementary in D.

If $s_1 \to C[y_1^+, y_2]$, the vertex s_1 has a negative neighbor on the path $C[y_2^+, y_1^-]$ and thus, D is cycle complementary.

Therefore we may assume that there exists a vertex $z_1 \in V(C[y_1^+, y_2^-])$ such that $z_1 \to s_1 \to C[z_1^+, y_2]$. We choose z_1 such that $C[z_1, y_2]$ has minimal length. Note that z_1 and y_2 are adjacent.

Subcase 2.3.2.4.3.3.1: If $z_1^+ \neq y_2$ and $y_2 \to z_1$, the vertex s_1 has a negative neighbor on the path $C[y_2^+, z_1^-]$ and thus, D is cycle complementary.

Subcase 2.3.2.4.3.3.2: If $z_1^+ \neq y_2$ and $z_1 \to y_2$, we are in Subcase 2.3.2.4.3.2 which we have already solved.

Subcase 2.3.2.4.3.3.3: Assume that $z_1^+ = y_2$. Note that y_1^+ has a negative neighbor w besides s_1 and y_1.

If $w \notin V(C[y_1^+, z_1])$, it is easy to check that D has complementary cycles.

If $w \in V(C[y_1^+, z_1])$, the vertex-disjoint cycles

$$C_1 = s_1y_2s_2C[y_2^+, y_1]x_1^1s_1 \quad \text{and} \quad C_2 = C[y_1^+, w]y_1^+$$

contain all vertices of D except $V(C[w^+, z_1])$. Note that if $z_1 \to C_1$, the digraph D is cycle complementary. By Lemma 2.1 there exists a vertex u on $C[w^+, z_1^-]$ such that the vertices of $C[u^+, z_1]$ can be inserted in C_2 (resulting in an extended cycle C_2') and $u \to C_2'$. It follows particularly that the vertex u^+ has a negative neighbor on C_2'. Now it is easy to check that D is cycle complementary.

For the opposite direction it is immediate that a 2-connected, 2-regular in-tournament with $2m+1$ ($m \geq 4$) vertices is not cycle complementary. □

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