A remark on degree sequences of multigraphs

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Abstract

A sequence $\{d_1, d_2, \ldots, d_n\}$ of nonnegative integers is *graphic* (*multigraphic*) if there exists a simple graph (multigraph) with vertices $v_1, v_2, \ldots, v_n$ such that the degree $d(v_i)$ of the vertex v_i equals d_i for each $i = 1, 2, \ldots, n$. The *(multi)graphic degree sequence problem* is: Given a sequence of nonnegative integers, determine whether it is (multi)graphic or not. In this paper we characterize sequences that are multigraphic in a similar way, Havel [4] and Hakimi [3] characterized graphic sequences. Results of Hakimi [3] and Butler [1] follow.

1 Notation and introduction

In this paper we consider finite, undirected graphs $G = (V, E)$ without loops with vertex set V and edge set E. A graph is called *simple* if there is at most one edge between each pair of vertices and a *multigraph* otherwise. The number of vertices $|V|$ is called the *order* of G and is denoted by $n(G)$.

If there is an edge between two vertices $u, v \in V$, then we denote the edge by uv. Furthermore, we call the vertex v a *neighbor* of u and say that uv is incident with u. The *neighborhood* of a vertex u is defined as the set $\{v \mid uv \in E\}$ and is usually denoted by $N(u)$. For a vertex $v \in V$ we define the *degree* of v as the number of edges incident with v. A vertex $v \in V$ is called *isolated* if $d(v) = 0$.

Let $X \subset V$ be a subset of the vertex set of a graph $G = (V, E)$. Then $G - X$ denotes the graph that is obtained by removing all vertices of X and all edges that are incident with at least one vertex of X from G. For a subset $Y \subset E$ of the edge set the graph $G - Y$ is obtained by removing all edges of Y.

A sequence $\{d_1, d_2, \ldots, d_n\}$ of nonnegative integers is *graphic* (*multigraphic*) if there exists a simple graph (multigraph) with vertices $v_1, v_2, \ldots, v_n$ such that $d(v_i) = d_i$ for each $i = 1, 2, \ldots, n$. Note that $d_i = 0$ for an $i \in \{1, 2, \ldots, n\}$ if and only if v_i is an isolated vertex. Therefore we only consider sequences $\{d_1, d_2, \ldots, d_n\}$, where $\min\{d_i \mid i = 1, 2, \ldots, n\} \geq 1$.

The *(multi)graphic degree sequence problem* is: Given a sequence of nonnegative integers, determine whether it is (multi)graphic or not. Havel [4] and Hakimi [3] presented a first solution of the graphic degree sequence problem in 1955 and 1962, respectively.

Theorem 1 (Havel [4] 1955, Hakimi [3] 1962)**.** *A sequence $d_1 \geq d_2 \geq \ldots \geq d_n$, where $n \geq 2$, of nonnegative integers is graphic if and only if the sequence $\{d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, d_{d_1+3}, \ldots, d_n\}$ is graphic.*

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We now turn our attention to the multigraphic degree sequence problem. A proof of the following characterization by Hakimi [3] can also be found in the article [6] of Takahashi, Imai and Asano.

Theorem 2 (Hakimi [3] 1962)**.** *A sequence $d_1 \geq d_2 \geq \ldots \geq d_n$, where $n \geq 2$, of nonnegative integers is multigraphic if and only if the sum $\sum_{i=1}^{n} d_i$ is even and $d_1 \leq d_2+d_3+\cdots+d_n$.*

In 1976, Boesch and Harary presented in their article [1] another solution which is due to Butler.

Theorem 3 (Butler [1] 1976)**.** *Let $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ be a sequence of nonnegative integers and let $2 \leq j \leq n$ be an index. Then the sequence $\{d_1, d_2, \ldots, d_n\}$ is multigraphic if and only if the sequence $\{d_1-1, d_2, d_3 \ldots, d_{j-1}, d_j-1, d_{j+1}, d_{j+2}, \ldots, d_n\}$ is multigraphic.*

This result suggests a construction method for multigraphs with a given degree sequence. Butler also proved that this procedure constructs a multigraph of maximal edge connectivity if the index j is selected equal to n in each step. In 1994, Takahashi, Imai and Asano [6] presented another algorithm to determine whether a given sequence of nonnegative integers is multigraphic and to construct a multigraph that realizes the degree sequence. However, the constructed multigraph is in general not connected (cf. Remark 9 and Figure 1).

Theorem 4 (Takahashi, Imai & Asano [6] 1994)**.** *Let $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ be a sequence of nonnegative integers and let $c = d_1 - d_2$. Then the sequence $\{d_1, d_2, \ldots, d_n\}$ is multigraphic if and only if the sequence $\{d'_1, d'_2, \ldots, d'_m\}$ is multigraphic, where the integers m and d'_j for $j = 1, 2, \ldots, m$ are defined as follows.*

(i) If $c \geq d_n$, then $m = n-1$, $d'_1 = d_1 - d_n$ and $d'_j = d_j$ for $j = 2, 3, \ldots, n-1$;

(ii) If $c = 0$, then $m = n-1$, $d'_j = d_j$ for $j = 1, 2, \ldots, n-2$ and $d'_{n-1} = d_{n-1} - d_n$;

(iii) If $0 < c < d_n$, then $m = n$, $d'_1 = d_1 - c$ and $d'_j = d_j$ for $j = 2, 3, \ldots, n-1$ and $d'_n = d_n - c$.

An overview of degree sequences and related problems can be found in [7].

In this paper we characterize sequences that are multigraphic in a way similar to the characterization of graphic sequences by Havel [4] and Hakimi [3] in Theorem 1. The results of Hakimi (Theorem 2) and Butler (Theorem 3) follow.

Theorem 5. *Let $n \geq 4$ be an integer and let $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ be a sequence of nonnegative integers. Let $2 \leq j \leq n$ be an index and let $1 \leq m \leq d_n$. Then the sequence $\{d_1, d_2, \ldots, d_n\}$ is multigraphic if and only if the sequence $\{d_1 - m, d_2, d_3 \ldots, d_{j-1}, d_j - m, d_{j+1}, d_{j+2}, \ldots, d_n\}$ is multigraphic.*

Choosing $j = n$ and $m = d_n$, we conclude the next result as a corollary of Theorem 5.

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Corollary 6. *Let $n \geq 4$ be an integer and let $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ be a sequence of nonnegative integers. Then the sequence $\{d_1, d_2, \ldots, d_n\}$ is multigraphic if and only if the sequence $\{d_1 - d_n, d_2, d_3 \ldots, d_{n-1}\}$ is multigraphic.*

Remark 7. *(a) Corollary 7 together with Observation 8 as the termination criterion leads to an algorithm that constructs a multigraph with a given degree sequence.*

(b) A necessary condition for a sequence $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ of integers to be realizable by a connected multigraph is that the sum of the integers satisfies the inequality $\sum_{i=1}^{n} d_i \geq 2n - 2$.

(c) Consider the algorithm suggested in (a). If $d'_1 = d'_{n-k} = r \geq 1$ in the k-th step of the construction, then $d'_1 = d'_2 = \ldots = d'_{n-k} = r$. A multigraph with this degree sequence is given by $G' = (V', E')$, where $V' = \{x_1, x_2, \ldots, x_{n-k}\}$ is the vertex set of G'. The edges of G' are defined as follows. The multigraph G' has $\frac{r}{2}$ edges between x_i and x_{i+1} if r is even and $\frac{r-1}{2}$ edges between x_{2j-1} and x_{2j} and $\frac{r+1}{2}$ edges between x_{2j} and x_{2j+1} if r is odd, where $i = 1, 2, \ldots, n-k$ and $j = 1, 2, \ldots, \frac{n-k}{2}$ and all indices are taken modulo $n-k$. Note that G' is r-edge-connected if $r \geq 2$.

(d) Let $\sum_{i=1}^{n} d_i \geq 2n - 2$ and consider the algorithm suggested by Corollary 7 and Observation 8. In each step of the construction of the multigraph two vertices are joined by j edges, where j is a sum (or difference) of some of the integers d_i. It follows that if (c) is used as an additional termination criterion, the multigraph constructed by the algorithm is p-edge-connected, where p is the greatest common divisor of $d_1, d_2, \ldots, d_n$.

(e) If two vertices are joined by j edges, this can be interpreted as a single weighted edge with weigth j. The weighted simple graph constructed by the procedure suggested in a) contains at most one cycle. Therefore it is a natural question to ask for necessary and sufficient conditions for a given degree sequence to be realizable by a weighted tree.

2 Results

Lemma 8. *Let $\mathbf{d} = d_1 \geq d_2 \geq \cdots \geq d_n$ be a non-increasing sequence of positive integers. If $\mathbf{d}$ is realizable by a 2-tree, then*

$$\sum_{i=1}^{k} \lceil d_i/2 \rceil \leq n + k - 2 \tag{1}$$

for every $k = 1, 2, \ldots, n$. Furthermore, if (1) is satisfied with equality, then for every 2-tree realization T of $\mathbf{d}$:

(i) the vertices corresponding to $d_1, d_2, \ldots, d_k$ induce a tree in T;

(ii) the vertices corresponding to $d_{k+1}, d_{k+2}, \ldots, d_n$ induce an independent set in T;

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(iii) every vertex of T is adjacent to at most one edge of weight 1.

Proof. Let T be a 2-tree with vertex set $\{x_1, x_2, \ldots, x_n\}$ such that $d(x_i) = d_i$. Since each edge has weight at most 2, it holds

$$\sum_{i=1}^{k} |N(x_i)| \geq \sum_{i=1}^{k} \left\lceil \frac{d_i}{2} \right\rceil .$$

On the other hand, the vertices $\{x_1, x_2, \ldots, x_k\}$ induce a weighted forest F in T and thus,

$$\sum_{i=1}^{k} |N(x_i)| = 2|E(F)| + |E(F, \overline{F})| \leq n + |E(F)| - 1 \leq n + k - 2.$$

... □

Lemma 9. *Let $\mathbf{d} = d_1 \geq d_2 \geq \cdots \geq d_n$ be a non-increasing sequence of positive integers. If ..., then $\mathbf{d}$ is not realizable by a 2-tree.*

Proof. ... □

Theorem 10. *Let $\mathbf{d} = d_1 \geq d_2 \geq \cdots \geq d_n$, where $n \geq 4$ be a non-increasing sequence of positive integers that is 2-realizable. Let $\mathbf{d}'$ be defined as follows:*

(i) If $d_n = 2$, let $\mathbf{d}'$ be a non-increasing ordering of $d_1 - 2, d_2, d_3, \ldots, d_{n-1}$.

(ii) If $d_n = 1$ and $d_1 - 1, d_2, d_3, \ldots, d_{n-1}$ is not *of the structure as described in Lemma 9, let $\mathbf{d}'$ be a non-increasing ordering of $d_1 - 1, d_2, d_3, \ldots, d_{n-1}$.*

(iii) If $d_n = 1$ and $d_1 - 1, d_2, d_3, \ldots, d_{n-1}$ is *of the structure as described in Lemma 9, let $\mathbf{d}'$ be a non-increasing ordering of $d_1, d_2, \ldots, d_{j-1}, d_j - 1, d_{j+1}, d_{j+2}, \ldots, d_{n-1}$, where j is the minimal index with the property that d_j is odd.*

The sequence $\mathbf{d}$ is realizable by a 2-tree if and only if $\mathbf{d}'$ is realizable by a 2-tree.

Proof. Clearly, if $\mathbf{d}'$ is realizable by a 2-tree, then the same holds for $\mathbf{d}$. Now suppose that $\mathbf{d}$ is 2-tree realizable. We shall discuss two cases depending on the value of d_n.

If $d_n = 2$, let $\mathbf{d}' = d'_1 \geq d'_2 \geq \cdots \geq d'_{n-1}$ be a non-increasing ordering of $d_1 - 2, d_2, d_3, \ldots, d_{n-1}$. Note that $\mathbf{d}'$ is not of the structure described in Lemma 9, since the original sequence $\mathbf{d}$ is not. Let T be a 2-tree with vertex set $\{x_1, x_2, \ldots, x_n\}$ and $d(x_i) = d_i$ chosen under the condition that the minimum j of the index set

$$\{i \colon x_i \text{ is adjacent to a leaf of } T\}$$

is minimal. If $j = 1$, then clearly $T - y$, where y is a leaf adjacent to x_1 is a 2-tree realization of $\mathbf{d}'$. So assume that $j \geq 2$.

□

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The proof of the above theorem immediately gives us a recursive construction rule to construct a 2-tree that has a given degree sequence.

Theorem 11. *Let* $\mathbf{d} = d_1 \geq d_2 \geq \cdots \geq d_n$ *be a non-increasing sequence of positive integers that is 2-realizable. The sequence is realizable by a 2-tree if and only if*

(1) for every $k = 1, 2, \ldots, n$*:* $\sum_{i=1}^{k} \lceil d_i/2 \rceil \leq n + k - 2$*;*

(2) ...

Proof. Necessity. This part follows from Lemmas 8 and 9.

Sufficiency. Suppose that $\mathbf{d}$ satisfies (1) and (2). The proof will be by induction on n. If $n = 2$, then either $d_1 = d_2 = 1$ or $d_1 = d_2 = 2$ both of which are realizable by a properly weighted K_2. If $n = 3$, then either $\mathbf{d} = 2, 1, 1$ or $\mathbf{d} = 3, 2, 1$ or $\mathbf{d} = 4, 2, 2$ all three of which are realizable by a path of length two with properly weighted edges.

Now let $n \geq 4$. We shall discuss two cases depending on whether a reduced sequence satisfies (2).

Suppose first that $d_1 - d_n, d_2, d_3, \ldots, d_{n-1}$ satisfies (2) and that $\lceil (d_1 - d_n)/2 \rceil < \lceil d_1/2 \rceil$. (In particular, this is the case when $d_n = 2$, since the original sequence $\mathbf{d}$ satisfies (2).) Let $\mathbf{d}' = d'_1 \geq d'_2 \geq \cdots \geq d'_{n-1}$ be the corresponding ordered sequence with $d'_r = d_1 - d_n$. Assume that $\mathbf{d}'$ does not satisfy (1). Let t be an index such that $\sum_{i=1}^{t} \lceil d'_i/2 \rceil > (n-1) + t - 2 = n + t - 3$. Obviously, it holds that $t < r$. Note that there exists a 2-tree realization of $\mathbf{d}$ such that x_n is a leaf. Hence there exists an index j such that the sequence $d_1, d_2, \ldots, d_{j-1}, d_j - d_n, d_{j+1}, d_{j+2}, \ldots, d_{n-1}$ is 2-tree realizable. Let $\mathbf{d}^* = d^*_1 \geq d^*_2 \geq \cdots \geq d^*_{n-1}$ be the corresponding ordered sequence with $d^*_s = d_j - d_n$. Clearly, $d^*_i \geq d'_i$ for $i = 1, 2, \ldots, j-1$. If $j \geq t+1$, then

$$\sum_{i=1}^{t} \lceil d^*_i/2 \rceil \geq \sum_{i=1}^{t} \lceil d'_i/2 \rceil > (n-1) + t - 2,$$

a contradiction to (1). If $j \leq t$, then $s \geq r > t$ and thus,

$$\sum_{i=1}^{t} \lceil d^*_i/2 \rceil = \sum_{i=1}^{t} \lceil d'_i/2 \rceil > (n-1) + t - 2,$$

again a contradiction to (1). So $\mathbf{d}'$ satisfies (1) and is realizable by a 2-tree by the induction hypothesis. Adding a vertex x and connecting x with x'_j via an edge of weight d_n results in a 2-tree realization of $\mathbf{d}$.

Suppose second that $d_1 - d_n, d_2, d_3, \ldots, d_{n-1}$ satisfies (2) and that $\lceil (d_1 - d_n)/2 \rceil = \lceil d_1/2 \rceil$. Then $d_n = 1$ and d_1 is even. Let $\mathbf{d}' = d'_1 \geq d'_2 \geq \cdots \geq d'_{n-1}$ be the corresponding ordered sequence with $d'_r = d_1 - 1$. Assume that $\mathbf{d}'$ does not satisfy (1). Let t be an index such that $\sum_{i=1}^{t} \lceil d'_i/2 \rceil > (n-1) + t - 2 = n + t - 3$. Since $\sum_{i=1}^{t} \lceil d'_i/2 \rceil = \sum_{i=1}^{t} \lceil d_i/2 \rceil$, it follows that $\sum_{i=1}^{t} \lceil d_i/2 \rceil = n + t - 2$. Hence any 2-tree realization of $\mathbf{d}$ has the structure described in Lemma 8.

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Suppose last that $d_1 - d_n, d_2, d_3, \ldots, d_{n-1}$ does not satisfy (2). Then $d_n = 1$, d_1 is odd, and $d_2, d_3, \ldots, d_{n-1}$ are even. We consider the sequence $d_1, d_2 - 1, d_3, d_4, \ldots, d_{n-1}$ which satisfies (2). Let $\mathbf{d}' = d'_1 \geq d'_2 \geq \cdots \geq d'_{n-1}$ be the corresponding ordered sequence with $d'_r = d_2 - 1$. Assume that $\mathbf{d}'$ does not satisfy (1). Let t be an index such that $\sum_{i=1}^{t} \lceil d'_i/2 \rceil > (n-1) + t - 2 = n + t - 3$. Obviously, it holds that $t < r$. Note that x_n is a leaf in every 2-tree realization of $\mathbf{d}$. Hence there exists an index j such that the sequence $d_1, d_2, \ldots, d_{j-1}, d_j - 1, d_{j+1}, d_{j+2}, \ldots, d_{n-1}$ is 2-tree realizable. $\square$

References

[1] F. Boesch and F. Harary, *Line removal algorithms for graphs and their degree lists*, IEEE Trans. Circuits and Systems **CAS-23** (1976), no. 12, 778–782, Special issue on large-scale networks and systems.

[2] P. Dankelmann and O. Oellermann, *Degree sequences of optimally edge-connected multigraphs*, Ars Combin. **77** (2005), 161–168.

[3] S. L. Hakimi, *On realizability of a set of integers as degrees of the vertices of a linear graph. I*, J. Soc. Indust. Appl. Math. **10** (1962), 496–506.

[4] V. Havel, *Eine Bemerkung über die Existenz der endlichen Graphen* (Czech), Časopis Pěst. Mat. **80** (1955), 477–480.

[5] D. Meierling and L. Volkmann, *A remark on degree sequences of multigraphs*, Math. Meth. Oper. Res. **69** (2009), 369–374.

[6] M. Takahashi, K. Imai, and T. Asano, *Graphical degree sequence problems*, IEICE Trans. on Fundamentals **E77-A** (1994), no. 3, 546–552.

[7] R. I. Tyshkevich, A. A. Chernyak, and Zh. A. Chernyak, *Graphs and degree sequences. I*, Kibernetika (Kiev) (1987), no. 6, 12–19, 133.

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